Addendum to “Optimal Risk Sharing with Different Reference Probabilities”: the Case of m Agents

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Abstract

We consider the problem of optimal risk sharing between m agents endowed with cash-invariant choice functions which are law-invariant with respect to different reference probability measures. As for the case of 2 agents considered in [1], we give sufficient conditions for the existence of Pareto optimal allocations in a discrete setting.

1 Setting and Formulation of the Problem

We consider a measurable space $(\Omega, \mathcal{F})$ and m probability measures $\mathbb{P}_1, ..., \mathbb{P}_m$ on $(\Omega, \mathcal{F})$ such that $(\Omega, \mathcal{F}, \mathbb{P}_i), i = 1, ..., m$ are non-atomic standard probability spaces. The measure $\mathbb{P}_i$ describes the view of agent i on the world $(\Omega, \mathcal{F})$ and $U_i : L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i) \to \mathbb{R}$ her preferences on $L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$. The choice function U_i is assumed to satisfy the following conditions:

(C1) concavity: $U_i(\alpha X + (1-\alpha)Y) \geq \alpha U_i(X) + (1-\alpha)U_i(Y)$ for all $X, Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ and $\alpha \in (0, 1)$;

(C2) cash-invariance: $U_i(X + c) = U_i(X) + c$ for all $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ and $c \in \mathbb{R}$;

(C3) normalization: $U_i(0) = 0$;

(C4) $\mathbb{P}_i$-law-invariance: $U_i(X) = U_i(Y)$ whenever $X, Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ are identically distributed under $\mathbb{P}_i$;

(C5) upper semi-continuity (u.s.c.): for any sequence $(X_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ converging to some $X \in L^\infty$, we have $U_i(X) \geq \limsup_n U_i(X_n)$.

We assume that the agents agree to exchange risk on a finite set of possible scenarios. Let $A = \{A_1, \ldots, A_n\} \subset \mathcal{F}$ be a finite partition of Ω and $\mathcal{F}_A := \sigma(\{A_1, \ldots, A_n\})$ the $\sigma-$algebra it generates. A is called admissible if

- $\mathbb{P}_i(A_j) > 0$ for all $j = 1, \ldots, n, i = 1, \ldots, m$,

- $\mathbb{P}_1(A_j) \in \mathbb{Q}^+$ for all $j = 1, \dots, n$.

The space of admissible financial positions which the agents consider in the exchange of risk, is the collection $\mathcal{S}_A$ of all $\mathcal{F}_A-$measurable random variables, that is isomorphic to $\mathbb{R}^n$. The optimal risk allocation problem, for any aggregate risk $X = \sum_{j=1}^n x_j 1_{A_j} \in \mathcal{S}_A$, is therefore formulated as follows:

$$\square_{i=1}^m u_i(x) = \sup_{\substack{y^1, ..., y^m \in \mathbb{R}^n, \\ y^1 + ... + y^m = x}} \sum_{i=1}^m u_i(y^i), \tag{1.1}$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $u_i(y_1^i, \dots, y_n^i) = U_i(\sum_{j=1}^n y_j^i 1_{A_j}), i = 1, ..., m$. We denote v_i the dual conjugate of u_i, $i = 1, ..., m$, and v the dual conjugate of $u = \square_{i=1}^m u_i$.

2 Existence result

Assumption 2.1. *Agents $2, ..., m$ give a finite penalty to the reference probability measure of agent 1, i.e.*

$$\mathbb{P}_1 \in dom(v_i), \ \forall i = 2, ..., m, \tag{2.1}$$

where $\mathbb{P}_1$ is identified with the vector $(p_1, \dots, p_n)$, with $p_j = \mathbb{P}_1(A_j)$ for all $j = 1, \dots, n$.

Assumption 2.2. *Either of the following two conditions holds:*

(i) No Risk-Arbitrage (NRA), i.e. $\square_{i=1}^m u_i(0) = 0$,

(ii) $\partial v_i(\mathbb{P}_1) \neq \emptyset$, $\forall i = 2, ..., m$.

Theorem 2.3. *Let $A = \{A_j\}_{j=1}^n$ be an admissible partition of Ω. Then, under Assumptions 2.1, 2.2, the convolution $\square_{i=1}^m u_i$ in (1.1) is exact at any $x \in \mathbb{R}^n$.*

In the proof of Theorem 2.3 we will use the following results.

Lemma 2.4. *(i) For all $x^1, ..., x^m \in \mathbb{R}^n$ such that $\sum_{i=1}^m x^i = 0$ and $x^i \in 0^+\mathcal{A}_i$, and for all $y \in dom(v)$, $\langle y, x^1 \rangle = 0$;*

(ii) $\square_{i=1}^m u_i$ is exact at every $x \in \mathbb{R}^n$ if and only if $\mathcal{A}_1 + ... + \mathcal{A}_m = \mathcal{A}_{\square_{i=1}^m u_i}$;

(iii) $\square_{i=1}^m u_i$ is exact at every $x \in \mathbb{R}^n$ if and only if $\mathcal{A}_1 + ... + \mathcal{A}_m$ is closed;

(iv) Let $C_1, ..., C_m$ be non-empty closed convex sets in $\mathbb{R}^n$. If there are no $x^1, ..., x^m$ not all zero in $\mathbb{R}^n$ such that $x^i \in 0^+C_i$ and $\sum_{i=1}^m x^i = 0$, then $C_1 + ... + C_m$ is closed.

Proof. (*of Theorem 2.3*) We first consider the case when the partition $A = \{A_1, \dots, A_n\}$ of Ω is balanced w.r.to $\mathbb{P}_1$, i.e. $\mathbb{P}_1(A_j) = \frac{1}{n}, \forall j = 1, \dots, n$. If $n = 1$, exactness of $\square_{i=1}^m u_i$ follows from cash-invariance. Henceforth, let $n \geq 2$. If there are no $x^1, ..., x^m$ not all zero in $\mathbb{R}^n$ such that $\sum_{i=1}^m x^i = 0$ and $x^i \in 0^+\mathcal{A}_i$, then the exactness follows from Lemma 2.4 (iii)-(iv). Now suppose there exist $x^1, ..., x^m$ not all zero in $\mathbb{R}^n$ such that $\sum_{i=1}^m x^i = 0$ and $x^i \in 0^+\mathcal{A}_i$. Define E on $\mathbb{R}^n$ by $E[z] = \frac{1}{n}\sum_{i=1}^n z_i$, and $\mathcal{E} = \{z \in \mathbb{R}^n : E[z] = 0\}$. From Assumption 2.1 and Lemma 2.4 (i), we have that $E[x^1] = 0$, hence $x^1 \in \mathcal{E} \cap 0^+\mathcal{A}_1$. Then we proceed as in the proof of Theorem 3.6 in [1] and obtain $u_1 = E$. Therefore $\Box_{i=1}^m u_i = E + v_2(\mathbb{P}_1) + ... + v_m(\mathbb{P}_1)$. Thus, if condition (i) of Assumption 2.2 holds, then $v_i(\mathbb{P}_1) = 0\,\forall i = 2, ..., m$ and $\Box_{i=1}^m u_i = E = u_1$, which in particular ensures the exactness of the convolution. On the other hand, if condition (ii) of Assumption 2.2 is satisfied, then for any $x \in \mathbb{R}^n$ and $y^i \in -\partial v_i(\mathbb{P}_1), i = 2, ..., m$, we have $y^1 := x - \sum_{i=2}^m y^i \in -\partial v_1(\mathbb{P}_1) = \mathbb{R}^n$. Therefore $(y^1, ..., y^m)$ is a POA of x, by Proposition 2.5, hence the convolution is exact.

Now consider a generic partition. By admissibility, the probabilities $a_i := \mathbb{P}_1(A_i)$ are in $\mathbb{Q}_+$ for all $i = 1, \ldots, n$. Consider the greatest rational number a s.t. a_i/a are all integers for $i = 1, \ldots, n$. By the non-atomicity of $(\Omega, \mathcal{F}, \mathbb{P}_k), k = 1, ..., m,$, for each $i = 1, \ldots, n$ we can find a partition $\{B_{i1}, \ldots, B_{im_i}\} \subset \mathcal{F}$ of the event A_i such that

$$\mathbb{P}_1(B_{ij}) = \frac{\mathbb{P}_1(A_i)}{m_i} = a \quad \text{and} \quad \mathbb{P}_k(B_{ij}) = \frac{\mathbb{P}_k(A_i)}{m_i}, k = 1, ..., m, \tag{2.2}$$

where $m_i := a_i/a$. Therefore, we end up with a $\mathbb{P}_1$-balanced admissible partition $B = \{B_{ij}, j = 1, \ldots, m_i, i = 1, \ldots, n\}$ of Ω, refinement of partition A, and we are back to the previous case (see the proof of Theorem 2.3 in [1]). □

Proposition 2.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $U_i : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to [-\infty, \infty)$ be proper concave u.s.c. functions, $i = 1, ..., m$. Then, for $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ s.t. $\partial \Box_{i=1}^m U_i(X) \neq \emptyset$ and for any allocation $(X_1, ..., X_m) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \times \cdots \times L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ of X,*

$$\Box_{i=1}^m U_i(X) = \sum_{i=1}^m U_i(X_i) \quad \Longleftrightarrow \quad \partial \Box_{i=1}^m U_i(X) = \cap_{i=1}^m \partial U_i(X_i).$$

References

[1] B. Acciaio and G. Svindland (2009). Optimal risk sharing with different reference probabilities, *Insurance: Mathematics and Economics* **44**, 426–433.