TWO PROBLEMS RELATED TO UTILITY THEORY UNDER UNUSUAL ASSUMPTIONS

I. OPTIMAL RISK SHARING with NON-MONOTONE CHOICE CRITERIONS

II. ABSOLUTELY-CONTINUOUS OPTIMAL MARTINGALE MEASURES

Beatrice Acciaio

– PhD Dissertation –

Perugia, January 26, 2006

Bibliography–Part I

[ADEH’99] P. Artzner, F. Delbaen, J.M. Eber, D. Heath (1999). “Coherent measures of risk”, *Mathematical Finance* **9**, 203-228.

[BEK’05] P. Barrieu, N. El Karoui (2005). “Inf-convolution of risk measures and optimal risk transfer”, *Finance and Stochastics* **9**, 269-298.

[FS’02] H. Föllmer, A. Schied (2002). “Convex measures of risk and trading constraints”, *Finance and Stochastics* **6/4**.

[JST’05] E. Jouini, W. Schachermayer, N. Touzi (2005). “Optimal risk sharing for law invariant monetary utility functions”, Preprint.

[MMRT’05] F. Maccheroni, M. Marinacci, A. Rustichini, M. Taboga (2005). “Portfolio selection with monotone mean-variance preferences”, Preprint.

ORS with Non-Monotone Choice Criterions

- Setting
- Class of Choice Criterions
- Definition of Sup-Convolution Problem
  - Existence and Characterization of the Solutions
- Constraints on Sup-Convolution Problem
  - Existence and Characterization of the Solutions
- Definition of Best-Monotone Approximation
  - Comparison Results Between Monotone and Non-Monotone Agents
- Explicit Solutions of Particular Sup-Convolution Problems

Setting

2 dates: today and tomorrow, $(\Omega, \mathcal{F}, \mathbb{P})$ non-atomic

financial positions: $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$

$\boldsymbol{n}$ agents: $(\xi_1, \ldots, \xi_n) \in L^\infty \times \cdots \times L^\infty$, $U_1, \ldots, U_n : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$

total risk $\boldsymbol{X} = \sum_{i=1}^{n} \xi_i \quad \Longrightarrow$ optimal sharing of $\boldsymbol{X}$

$\swarrow \qquad \searrow$

collective | individual

Monetary Utility Functionals

Def. $U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ **m.u.f.** satisfies the following properties:

- concavity: $U(\alpha X + (1-\alpha)Y) \geq \alpha U(X) + (1-\alpha)U(Y)$, $\forall \alpha \in [0,1]$,
- monotonicity: $U(X) \geq U(Y)$, whenever $X \geq Y$,
- cash-invariance: $U(X + c) = U(X) + c$, $\forall c \in \mathbb{R}$,
- normalization: $U(0) = 0$

$\Longrightarrow$ U is finite and Lipschitz-continuous on L^∞

Remark: $\rho(X) = -U(X)$ **convex risk measure** ([FS'02])

Examples of m.u.f.

- **Average Value at Risk** ([ADEH'99])

$$U_\lambda(X) := -AV@R_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda q_X(t)dt = \inf_{Z\in[0,\frac{1}{\lambda}]} \mathbb{E}[ZX], \quad \lambda \in (0,1)$$

where $q_X(t) = \inf\{x : \mathbb{P}(X \le x) > t\} = -V@R_t(X)$

- **Semi-Deviation Utility**

$$U_\delta^p(X) := \mathbb{E}[X] - \delta\mathbb{E}[(X - \mathbb{E}[X])_-^p]^{1/p}, \quad 0 < \delta \le 1, \ 1 \le p \le \infty$$

- **Entropic Utility**

$$U_\gamma^{en}(X) := -\gamma \ln \mathbb{E}\left[\mathrm{e}^{-\frac{X}{\gamma}}\right] = \sup\{m \in \mathbb{R} : \mathbb{E}[u(X-m)] \ge u(0)\},$$

where $u(x) = -\mathrm{e}^{-x/\gamma}, \ \gamma > 0$

Lack of Monotonicity

Assumption [A] $U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ satisfies the following properties:

- concavity: $U(\alpha X + (1-\alpha)Y) \geq \alpha U(X) + (1-\alpha)U(Y)$, $\forall \alpha \in [0,1]$,
- cash-invariance: $U(X + c) = U(X) + c$, $\forall c \in \mathbb{R}$,
- normalization: $U(0) = 0$
- $\| \,.\, \|_\infty$ – continuity

▶ **Mean-Variance Principle**

$U_\delta^{mv}(X) := \mathbb{E}[X] - \delta Var(X), \;\; \delta > 0$

▶ **Standard-Deviation Principle**

$U_\delta^{sd}(X) := \mathbb{E}[X] - \delta \sqrt{Var(X)}, \;\; \delta > 0$

Toy Example: Lottery Ticket

states of the world	ω_1	ω_2
probabilities	α	$1-\alpha$
prospect X	0	0
prospect Y	0	y

with $\alpha \in (0,1)$ and $y \in \mathbb{R}^+ = (0, +\infty)$.

$$U_\delta^{mv}(Y) < 0 = U_\delta^{mv}(X)$$

whenever $y > \dfrac{1}{\delta\alpha}$

$\Rightarrow$

mean-variance agent refuses the gift of a rich-lottery ticket

Optimization: the Collective Point of View

Def. Set of **attainable allocations**:

$$\mathbb{A}_n(X) = \left\{ (X_1, \dots, X_n) \in L^\infty \times \cdots \times L^\infty : \sum_{i=1}^n X_i = X \right\}$$

Def. Set of **attainable allocations increasing with the aggregate risk**:

$$\mathbb{A}_n^\uparrow(X) = \{(X_1, \dots, X_n) \in \mathbb{A}_n(X) : X_i = \phi_i(X), \text{ with } \phi_i \text{ non-decreasing}\}$$

(SC)

$$\boldsymbol{U_1 \square ... \square U_n(X) := \sup_{(X_1,..,X_n) \in \mathbb{A}_n(X)} \sum_{i=1}^n U_i(X_i)}$$

Pareto Optimal Allocations

Def. An allocation $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ is said **Pareto Optimal** if:
for any $(Y_1, \ldots, Y_n) \in \mathbb{A}_n(X)$ such that $U_i(Y_i) \geq U_i(X_i)\ \forall\, i = 1, \ldots, n$
$\Rightarrow\ U_i(Y_i) = U_i(X_i)\ \forall\, i = 1, \ldots, n$.

Remark: POA are defined up to constants summing up to zero:
for any $(X_1, \ldots, X_n)$ POA, $c_1, \ldots, c_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} c_i = 0$
$\Rightarrow\ (X_1 + c_1, \ldots, X_n + c_n)$ POA.

- It may happen that some agent suffers a loss in changing her position from the initial one to a POA.

Characterization of POA

THEOREM. Let $(U_i)_{i=1}^n$ satisfy [A] and $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$. Then TFAE:

(i) $(X_1, \ldots, X_n)$ is a POA of X

(ii) $\sum_{i=1}^n U_i(X_i) = U_1 \Box \cdots \Box U_n(X)$

(iii) $\exists \mu \in (L^\infty)^*$ s.t. $U_i(X_i) = V_i(\mu) + \langle \mu, X_i \rangle \; \forall \, i = 1, \ldots, n$

where $V_i : (L^\infty)^* \to [0, +\infty]$ is the **penalty function** associated to U_i:

$$V_i(\mu) = \sup_{X \in L^\infty} \{U_i(X) - \langle \mu, X \rangle\}$$

(see [JST'05] for the case of two m.u.f.)

Some Convex Analysis

$U : L^\infty \to \mathbb{R}$ proper, concave, continuous

$V : (L^\infty)^* \to [0, +\infty]$ convex conjugate function

- $V(\mu) = \sup_{X \in L^\infty} \{U(X) - \langle \mu, X \rangle\}$
- $U(X) = \inf_{\mu \in (L^\infty)^*} \{V(\mu) + \langle \mu, X \rangle\}$

$\Rightarrow$ U and V are $\langle L^\infty, (L^\infty)^* \rangle$−conjugate

$\partial V(\mu) = \{X \in L^\infty : V(\eta) \geq V(\mu) + \langle \eta - \mu, X \rangle, \forall \eta \in (L^\infty)^*\}$

$\partial U(X) = \{\mu \in (L^\infty)^* : U(Y) \leq U(X) + \langle \mu, Y - X \rangle, \forall Y \in L^\infty\}$

$$\mu \in \partial U(X) \iff X \in -\partial V(\mu) \iff U(X) = V(\mu) + \langle \mu, X \rangle$$

Existence of POA

Def. U is said **law-invariant** if : $X =_d Y \ \Rightarrow\ U(X) = U(Y)$.

THEOREM. Let $(U_i)_{i=1}^n$ be law-invariant and satisfying [A].
Then, for any aggregate risk $X \in L^\infty$,

$$\sup_{(X_1,..,X_n)\in \mathbb{A}_n(X)} \sum_{i=1}^n U_i(X_i) \;=\; \sup_{(X_1,..,X_n)\in \mathbb{A}_n^{\uparrow}(X)} \sum_{i=1}^n U_i(X_i),$$

and the set of POAs in $\mathbb{A}_n^{\uparrow}(X)$ is non-empty.

(see [JST'05] for the case of two m.u.f.)

Optimization: the Individual Point of View

Initial risk endowment $(\xi_i)_{i=1}^n \longrightarrow$ aggregate risk $X = \sum_{i=1}^n \xi_i$

Def. $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$ satisfies the **Individual Rationality** constraints if:

(IR) $U_i(X_i) \geq U_i(\xi_i), \ \forall i = 1, .., n$

Def. $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$ is an **Optimal Risk Sharing** rule if it is Pareto optimal and satisfies the individual rationality constraints:

ORS = POA + (IR)

$\Rightarrow$ **(CSC)** $\begin{cases} \sup_{(X_i)_{i=1}^n \in \mathbb{A}_n(X)} \sum_{i=1}^n U_i(X_i) \\ U_i(X_i) \geq U_i(\xi_i), \ i = 1, .., n \end{cases}$

Characterization of ORS Rules

THEOREM. Let $(U_i)_{i=1}^n$ satisfy [A] and $(X_i)_{i=1}^n$ be a POA of $X = \sum_{i=1}^n \xi_i$. Define the **indifference prices** $\boldsymbol{p_i} := U_i(X_i) - U_i(\xi_i),\ i = 1, ..., n$. Then:

(i) $\sum_{i=1}^n p_i \geq 0$;

(ii) $\sum_{i=1}^n \pi_i = 0,\ (X_i - \pi_i)_{i=1}^n$ ORS rule $\iff$ $\boldsymbol{\pi_i} \leq \boldsymbol{p_i} \quad \forall i = 1, ..., n$

(see [JST'05] for the case of two m.u.f.)

Set of **suitable prices** for the POA $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$:

$$\mathbf{\Pi} := \left\{(\pi_i)_{1=1}^n \in \mathbb{R}^n : \sum_{i=1}^n \pi_i = 0,\ \pi_i \leq p_i, \forall i = 1, ..., n\right\}$$

Recipe to Find the Optimal Contract

PARETO OPTIMALITY	$\longrightarrow$	**SHARING** of the total risk
INDIVIDUAL RATIONALITY	$\longrightarrow$	**SUITABLE PRICES** of the contract
MARKET POWER of the agents	$\longrightarrow$	**THE PRICE** of the contract

Example. Ultimatum game ([BEK'05]). The best price for agent i:

$$(P_i) \qquad \begin{cases} \sup\limits_{X_j, \pi_j, j \neq i} U_i\Big(X - \sum\limits_{j \neq i} (X_j - \pi_j)\Big) \\ U_j(X_j - \pi_j) \geq U_j(\xi_j), \ \forall j \neq i. \end{cases}$$

Non-monotone vs Monotone Criterions

Def. **Domain of monotonicity** of U:

$$\mathbf{M}(\mathbf{U}) := \{X \in L^\infty : \partial U(X) \cap (L^\infty)^*_+ \neq \emptyset\}$$

Def. The **best monotone-approximation** U^{m} of U ([MMRT'05]):

$$\mathbf{U}^{\mathrm{m}}(\mathbf{X}) := \sup\{U(Y), Y \in L^\infty, Y \leq X\} = U \Box U_{worst}(X)$$

$\Rightarrow$ U^{m} is the most conservative m.u.f. that dominates U:

$$\mathbf{U}^{\mathrm{m}}(\mathbf{X}) \begin{cases} = U(X), & \text{if } X \in M(U), \\ > U(X), & \text{otherwise} \end{cases}$$

Example: Mean-Variance Agent

Mean-variance preferences: $\mathbf{U}_\delta^{\mathbf{mv}}(\mathbf{X}) = \mathbb{E}[X] - \delta Var(X), \;\; \delta > 0$

Domain of monotonicity: $\mathbf{M}(\mathbf{U}_\delta^{\mathbf{mv}}) = \left\{ X \in L^\infty : X - \mathbb{E}[X] \leq \frac{1}{2\delta} \right\}$

$$\mathbf{U}_\delta^{\mathbf{mmv}}(\mathbf{X}) = \begin{cases} U_\delta^{mv}(X), & \text{if } X \in M(U_\delta^{mv}), \\ U_\delta^{mv}(X \wedge k_X), & \text{else}, \end{cases}$$

where $k_X = \max\{t \in \mathbb{R} : X \wedge t \in M(U_\delta^{mv})\}$

$\longrightarrow$ truncation of the prospect from above

$\longrightarrow$ on the toy example we bound the winnings

On the Toy Example

states of the world	ω_1	ω_2
probabilities	α	$1-\alpha$
prospect X	0	0
prospect Y	0	y

$$U_\delta^{\boldsymbol{mv}}(Y) < 0, \ \forall y > \frac{1}{\delta\alpha}$$

$\Rightarrow$ mv-agent refuses a rich-lottery ticket

$$U_\delta^{\boldsymbol{mmv}}(Y) > 0, \ \forall y \in \mathbb{R}^+$$

threshold $k_Y = \dfrac{1}{2\delta\alpha}$

$\Rightarrow$ monotone mv-agent accepts any lottery-ticket

POA and Monotonicity

- $U_1, \ldots, U_n$ satisfy Assumption [A], at least one monotone (m.u.f.)

$$(\boldsymbol{SC}) \qquad U(X) := \sup_{(X_i)_i \in \mathbb{A}_n(X)} \sum_{i=1}^{n} U_i(X_i)$$

$$(\boldsymbol{SC^m}) \qquad W(X) := \sup_{(X_i)_i \in \mathbb{A}_n(X)} \sum_{i=1}^{n} U_i^{\mathrm{m}}(X_i)$$

$U \equiv W$ on L^∞ and $\{\text{sol. of } (SC)\} \subseteq \{\text{sol. of } (SC^m)\}$

at least one strictly monotone,
non-monotone ones of mv-type $\Rightarrow$ $\{\text{sol. of } (SC)\} \equiv \{\text{sol. of } (SC^m)\}$

ORS and Monotonicity

- $U_1, \ldots, U_n$ satisfy Assumption [A], at least one monotone (m.u.f.):

  for any POA, indifference price w.r.t. $U_i \geq$ indifference price w.r.t. $U_i^{\mathbf{m}}$

  $\Rightarrow$ suitable prices w.r.t. $(U_1^{\mathbf{m}}, .., U_n^{\mathbf{m}})$ are suitable prices w.r.t. $(U_1, .., U_n)$

- If at least one is strictly monotone (e.g. entropic, semi-deviation with $p \neq +\infty$) and the non-monotone ones are of mean-variance type, then

$$\{\text{ORS for } (U_1^{\mathbf{m}}, .., U_n^{\mathbf{m}})\} \subseteq \{\text{ORS for } (U_1, .., U_n)\}$$

Entropic vs Mean-Variance vs Standard-Deviation

$$U_1(X) := U_\gamma^{en}(X) = -\gamma \ln \mathbb{E}[\mathrm{e}^{-\frac{X}{\gamma}}], \quad \gamma > 0$$

$$U_2(X) := U_{\delta_1}^{mv}(X) = \mathbb{E}[X] - \delta_1 Var(X), \ \delta_1 > 0$$

$$U_3(X) := U_{\delta_2}^{sd}(X) = \mathbb{E}[X] - \delta_2 \sqrt{Var(X)}, \ \delta_2 > 0$$

$U_1 \Box U_2 \Box U_3(X) \Rightarrow$

> $\exists$ a unique (up to const) POA (X_1, X_2, X_3) of X, with X_1 (resp. X_2, X_3) convex (resp. concave) function of the total risk X, and s.t. $X_2 = \alpha X_3$.

- $U_{\delta_1}^{mv}-$agent and $U_{\delta_2}^{sd}-$agent especially take the worst risks, whereas $U_\gamma^{en}-$agent especially takes the lowest risks (prudent).

Average Value at Risk vs (S) - agents

Def. U is said strictly risk-averse conditionally on any event if it satisfies

(S) $U(X) < U(X1_{A^c} + \mathbb{E}[X|A]1_A)$, $\forall X \in L^\infty$ and $A \in \mathcal{F}$: $X|_A \not\equiv$const

$$U_1(X) := -AV@R_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda q_X(t)dt, \ \lambda \in (0,1)$$

U_2 : satisfying (S)+(li) (e.g. entropic, mean-variance, standard-deviation)

$U_1 \square U_2(X) \Rightarrow$

> There exists a unique (up to a const) POA (X_1, X_2) of X:
> $$(X_1, X_2) = (-(X-l)^- + (X-u)^+, (l \vee X) \wedge u), \ \ l, u \in \mathbb{R}$$

- $AV@R-$agent takes the extreme risks. They exchange at the most two European options written on X (limited stop-loss contract).

Examples of AV@R-agents vs (S)-agents

$U_\lambda \Box U_\delta^{mv}(X) \Rightarrow$

If $(esssup X - essinf X) < \frac{1}{2\delta} \wedge \frac{1}{2\delta}\Big(\frac{1}{\lambda} - 1\Big)$, the unique POA of X is $(X_1, X_2) = (0, X)$ (full-insurance contract).

$U_\lambda \Box U_\gamma^{en} \Box U_\delta^{mv}(X) \Rightarrow$

$\exists$ a unique POA (X_1, X_2, X_3): $X_1 = -(X-k)^-$ and X_2(resp. X_3) convex(resp. concave) function of $X \vee k$.

$U_\lambda \Box U_{\delta_1}^{mv} \Box U_{\delta_2}^{sd}(X) \Rightarrow$

$\exists$ a unique (up to const) POA (X_1, X_2, X_3) of X:
$(-(X-l)^- + (X-u)^+, \alpha(X - X_1), (1-\alpha)(X - X_1))$.

Mean-Variance vs Standard-Deviation

$$U^{mv}_{\delta_1}(X) = \mathbb{E}[X] - \delta_1 Var(X),\ \delta_1 > 0$$

$$U^{sd}_{\delta_2}(X) = \mathbb{E}[X] - \delta_2 \sqrt{Var(X)},\ \delta_2 > 0$$

$U^{mv}_{\delta_1}$ vs $U^{sd}_{\delta_2}$ $\Rightarrow$

> There exists a unique (up to a const) POA (X_1, X_2) of X
> $(X_1, X_2) = (\alpha X, (1-\alpha)X)$ (quota-share contract)

where $$\alpha = \begin{cases} \dfrac{\delta_2}{2\delta_1\sqrt{Var(X)}}, & \text{if } \sqrt{Var(X)} \geq \dfrac{\delta_2}{2\delta_1}, \\ 1, & \text{otherwise} \end{cases}$$

$\longrightarrow$ Proportional sharing of the total risk

Semi-Deviation vs Standard-Deviation

$$U_\delta^2(X) = \mathbb{E}[X] - \delta \mathbb{E}[(X - \mathbb{E}[X])_-^2]^{1/2}, \; 0 < \delta \leq 1$$

$$U_\delta^{sd}(X) = \mathbb{E}[X] - \delta \sqrt{Var(X)}, \; 0 < \delta \leq 1$$

Remark. $U_\delta^{sd}(X) \leq U_\delta^2(X), \; \forall X \in L^\infty$

U_δ^2 vs U_δ^{sd} $\Rightarrow$

U_δ^2-agent takes the total risk X: set of POAs $= \{(X - c, c), \; c \in \mathbb{R}\}$

On the Comonotonicity Property

Def. U is said **comonotone** if $U(X+Y) = U(X) + U(Y)$ for any pair (X, Y) of comonotone r.v.'s: $(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0$ a.s.

THEOREM. Let $(U_i)_{i=1}^n$ be comonotone, law-invariant m.u.f.'s. Then any POA is obtained as exchange among the agents of European options ("stop-loss contracts") written on the aggregate risk X.

Bibliography–Part II

[BF'02] F. Bellini, M. Frittelli (2002). "On the existence of minimax martingale measures", *Mathematical Finance* **12/1**, 1-21.

[S'01] W. Schachermayer (2001). "Optimal investment in incomplete markets when wealth may become negative", *Annals of Applied Probability* **11/3**, 694-734.

Absolutely Continuous Optimal Martingale Measures

- Setting
- Primal Problem
- Assumptions
- Dual Problem
  - Equivalent Case
  - Absolutely-Continuous Case
- Auxiliary Problems
- Convergence Results
- The Absolutely-Continuous Case: an Example

Setting

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, time horizon $T \in (0, +\infty]$
- $S = (S_t)_{0\le t\le T}$: $\mathbb{R}^d$-valued locally-bounded semimartingale process (discounted price process of d traded assets)
- $\mathcal{H} := \{H : H$ predictable, $S-$integrable and s.t. $(H \cdot S) \ge -c\}$ (admisible trading strategies)
- $u : \mathbb{R} \to \mathbb{R}$ utility function: smooth, strictly increasing, strictly concave
- $x \in \mathbb{R}$: initial endowment of the economic agent

$$(\boldsymbol{PP}) \qquad \boldsymbol{w(x) = \sup_{H\in\mathcal{H}} \mathbb{E}[u(x + (H \cdot S)_T)]}$$

Assumptions

★ (NFLVR): $\mathcal{M}^e(S) := \{\mathbb{Q} \sim \mathbb{P} : \ S$ is a $\mathbb{Q}$- local martingale$\} \neq \emptyset$

★ Inada conditions: $\lim_{x \to -\infty} u'(x) = \infty$ and $\lim_{x \to +\infty} u'(x) = 0$

★ (RAE) conditions: $\liminf_{x \to -\infty} \frac{xu'(x)}{u(x)} > 1$ and $\limsup_{x \to +\infty} \frac{xu'(x)}{u(x)} < 1$

★ $\sup_{H = H1_{]\rho, T]}} \mathbb{E}[u((H \cdot S)_T) | \mathcal{F}_\rho] < u(\infty)$ a.s. $\forall \rho \in [0, T]$ stopping time

★ Every stopping time is $(\mathcal{F}_t)-$predictable

Dual Formulation of the Problem

$v : \mathbb{R}^+ \to \mathbb{R}$ convex conjugate of u : $v(y) = \sup_{x \in \mathbb{R}} (u(x) - xy)$

$$(\boldsymbol{DP}) \qquad \boldsymbol{\nu(y) = \inf_{\mathbb{Q} \in \mathcal{M}^a(S)} \mathbb{E}\left[v\left(y \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]}$$

where $\mathcal{M}^a(S) := \{\mathbb{Q} \ll \mathbb{P} : \ S \text{ is a } \mathbb{Q} - \text{local martingale}\}$

Example: $u(x) = -e^{-x} \longrightarrow v(y) = y(\ln y - 1)$. In this case, to solve (DP) is equivalent to minimize the relative entropy $H(\mathbb{Q};\mathbb{P})$.

▶ PROCEDURE : solve $(\boldsymbol{DP}) \longrightarrow$ convex duality $\longrightarrow$ solve $(\boldsymbol{PP})$

From the Dual to the Primal Problem

$$\widehat{X}_T(x) \in \mathcal{F}_T$$

optimal terminal wealth, solution to $(\boldsymbol{PP})$
(it exists unique $\forall\, x \in \mathbb{R}$, [S'01])

$$\widehat{\mathbb{Q}}_y \in \mathcal{M}^a(S)$$

minimax martingale measure, solution to $(\boldsymbol{DP})$
(it exists unique $\forall\, y > 0$, [BF'02])

$\Longrightarrow$

$$\frac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}} = \frac{u'(\widehat{X}_T(x))}{y}$$

where $y = w'(x) > 0$ ([S'01])

On the Minimal Martingale Measure

EQUIVALENT CASE: $\boxed{\widehat{\mathbb{Q}} \sim \mathbb{P}}$ $\longrightarrow$ $\widehat{X}_T(x) = x + (\widehat{H}(x) \cdot S)_T$

NON-EQUIVALENT CASE: $\boxed{\widehat{\mathbb{Q}} \ll \mathbb{P}}$ $\longrightarrow$ **?**

$\downarrow$

By means of auxiliary problems: $\widehat{X}_T(x) = x + \mathbb{P}\text{-}\lim_{n \to +\infty} (H^n(x) \cdot S)_T$

Auxiliary Problems

Density process corresponding to $\widehat{\mathbb{Q}}$: $\quad \boldsymbol{Z_T} := \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}$

$$\boldsymbol{Z_t} := \mathbb{E}[Z_T|\mathcal{F}_t], \quad \forall t \in [0,T)$$

Stopping times: $\quad \boldsymbol{\tau} := \inf\{t > 0 : Z_t = 0\}$

$$\boldsymbol{\tau_n} := \inf\{t > 0 : Z_t \leq \tfrac{1}{n}\} \uparrow\uparrow \boldsymbol{\tau}$$

$$\boldsymbol{(DP_n)} \quad \boldsymbol{\nu_n(y)} = \inf_{\mathbb{Q}\in\mathcal{M}^a(S^{\tau_n})} \mathbb{E}\left[v\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \quad \longrightarrow \quad \mathbb{Q}^{(n)} \sim \mathbb{P}$$

$$\Downarrow$$

$$\boldsymbol{(PP_n)} \quad \boldsymbol{w_n(x)} = \sup_{H\in\mathcal{H}} \mathbb{E}[u(x+(H\cdot S)_{\tau_n})] \quad \longrightarrow \quad X_T^{(n)}(x) = x+(H^n\cdot S)_T$$

Convergence Results

THEOREM. Under our assumptions:

$i)\ w_n(x) \xrightarrow[n]{} w(x)$

$ii)\ X_T^{(n)}(x) \xrightarrow[n]{\mathbb{P}} \widehat{X}_T(x)$ and $\dfrac{d\mathbb{Q}_{y_n}^{(n)}}{d\mathbb{P}} \xrightarrow[n]{L^1(\mathbb{P})} \dfrac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}$

where $y = w'(x)$ and $y_n = w_n'(x)$

$$\Longrightarrow \quad \widehat{X}_T(x) = x + \mathbb{P}\text{-}\lim_{n \to +\infty} (H^n(x) \cdot S)_T$$

Example: a Trinomial-Tree Model

- $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$, $\Omega = C_0 = C_n \cup (\cup_{k=1}^{n}(A_k \cup B_k)) = C_\infty \cup (\cup_{k=1}^{\infty}(A_k \cup B_k))$

- $(S_n)_{n\in\mathbb{N}}$ trinomial tree : at each step $\begin{array}{ll} \nearrow & \text{go up} \\ \rightarrow & \text{remain at the same level} \\ \searrow & \text{go down} \end{array}$

$S_0 = 0 = x$ and $S_n = \begin{cases} a_k, & \text{on } A_k,\ k = 1,..,n \\ b_k, & \text{on } B_k,\ k = 1,..,n \\ a_{n+1}, & \text{on } C_n \end{cases}$

- where sgn(a_n) =sgn$(b_n) = (-1)^n$; $|a_n| \ll |b_n|$; $|a_n|, |b_n| \nearrow +\infty$

Example: a Trinomial-Tree Model

Optimal terminal wealth : $\widehat{X}_\infty(0) = \begin{cases} (1 \cdot S)_\infty = \lim_n S_n, & \text{where it exists,} \\ +\infty, & \text{otherwise} \end{cases}$

$$= \begin{cases} a_n, & \text{on } A_n, \ n \in \mathbb{N} \\ b_n, & \text{on } B_n, \ n \in \mathbb{N} \\ +\infty, & \text{on } C_\infty, \text{ with } \mathbb{P}(C_\infty) > 0 \end{cases}$$

Optimal martingale measure : $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \propto u'(\widehat{X}_\infty(0))$

$\mathbb{P}\Big(\{\widehat{X}_\infty = +\infty\}\Big) > 0$ and $\widehat{\mathbb{Q}} \ll \mathbb{P}$