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Tesi di Dottorato

Due problemi relativi alla teoria dell'utilità in condizioni inusuali

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To Marco, Giancarlo and Assunta,
to all those persons on which
I have always relied,
to those who have been important
but are no longer here,
and to those who will be important
but I have yet to meet.

"E quindi uscimmo a riveder le stelle"
Dante, *Divina Commedia* Inf. XXXIV, 139

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Chapter 1

Introduction

The utility theory suggests that, in case of risk or uncertainty, the decision maker relies on her preferences when choosing. In particular, a compelling, widely applied way to represent preference relations is via the expected utility. A classical problem of financial economics is based on this ground: an agent invests her initial capital in a security market with the aim of maximizing the expected utility of her terminal wealth. In the context of continuous-time models, this problem was first studied by Merton [61, 62]. Therefore, the introduction in the Eighties of the notion of equivalent martingale measures, created the possibility for a new approach to portfolio management by martingale duality methods (see Pliska [65], Karatzas et al. [52], Cox and Huang [14, 15] for the case of complete markets; He and Pearson [43, 44], Karatzas et al. [53] in incomplete markets, as well as the more recent works of Kramkov and Schachermayer [56, 57], Schachermayer [69], Biagini and Frittelli [7], among others). Moreover, the powerful tool of the expected utility also leads to different answers to the problem of pricing and hedging contingent claims in incomplete markets (see, e.g., Föllmer and Schweizer [31], Delbaen et al. [23], El Karoui and Rouge [27], Frittelli [34], Hugonnier et al. [46], Musiela and Zariphopoulou [63], Davis [19], among many others) as well as to applications in the framework of the risk exchange theory (see, e.g., Kaluszka [51], Dana and Scarsini[18], and the references therein).

It was only at the end of the Nineties that, due to the pioneering work of Artzner et al. [3], a new axiomatic approach to the measures of risk/utility was developed, in order to quantify the riskiness of positions as a capital requirement (see, e.g., Delbaen

[20, 21], Föllmer and Schied [32, 30, 33], Frittelli and Rosazza Gianin [35, 36, 37]). These new measures also allow applications within the framework of pricing/hedging (see, e.g., Delbaen [22], Barrieu and El Karoui [5], Klöppel and Schweizer [54], Xu [72]) and optimal risk sharing (see, e.g., Barrieu and El Karoui [4, 5], Jouini et al. [49]).

In this thesis we consider two distinct problems, both concerning the utility theory, where some usual assumptions are weakened. In Part I (Chapters 3-5) we study the problem of optimal sharing of risk among several economic agents, using the above-mentioned new axiomatic approach. For example, this problem arises when an agent wants to insure a certain risky position and therefore aims at sharing her initial exposition with an insurer; or in the case of the sharing of health costs between the patient and the hospital; or when a company must assign liabilities to its daughter companies. We consider the generic situation of an aggregate of n economic agents, for any $n \geq 2$, which agree on exchanging risks, provided that each of them improves her own level of satisfaction by passing from the initial position to the new one. In order to formalize that, we regard each agent as characterized by an initial risk endowment ξ_i (representing her future financial position) and a choice functional U_i (modelling her preferences and defined on the space of all admissible financial positions). In formulating and solving our optimization problems we are strongly inspired by the recent work of Jouini, Schachermayer and Touzi [49], where the considered choice functionals are, up to the sign, convex risk measures in the sense of Föllmer and Schied [33]. In particular, in the paper cited above, concavity, monotonicity and translation-invariance are all required on the involved choice criterions, whereas here we weaken these conditions, allowing these functionals to be non-monotone. This will lead us to include, in our study, well known preference criterions as the mean-variance and the standard-deviation principles (see the seminal paper of Markovitz [60] for the portfolio selection problem, and Bühlmann [11, §4] for the premium calculation problem). On the other hand, in this way we incur the drawback of violating one of the most acknowledged economic principles, expressed as "more is preferred to less". This means that pathological situations may occur, in the sense that a certain payoff can be preferred to a bigger one. Nevertheless, with regard to the problem of re-sharing the aggregate risk $X = \sum_{i=1}^{n} \xi_i$ among the agents, we provide results of the existence and the characterization of optimal solutions analogous to those given in [49]. In particular, our interest moves in two directions: firstly, we aim at maximizing the overall utility (common welfare); secondly, we do not want any of the agents to suffer a loss in changing her financial position (individual incentive). This yields a constrained optimization problem, which consists in the sup-convolution of the involved choice functionals, under side constraints depending on the initial risk endowment of the agents (*individual rationality* (IR) constraints). At this point the property of translation-invariance induces a remarkable simplification, as stressed in [49]. Indeed, we can first solve the unconstrained optimization problem, which produces the *Pareto optimal allocations* (POAs) of the total risk X. In other words, by doing so we obtain such redistributions of X that do not allow any other allocation which is better for each agent. Afterwards, we impose the IR constraints which select, among all the POAs, those that make each agent willing to enter into the contract. Note that, for every Pareto optimal allocation $(X_1, \ldots, X_n)$ we can arbitrarily rebalance the cash without changing the aggregate utility $\sum_{i=1}^{n} U_i(X_i)$. Therefore, the solutions to the unconstrained sup-convolution problem define how to re-share the total risk, but not the (right) price for this exchange. On the other hand, by imposing the individual rationality constraints, a set of suitable prices is singled out. In this way we have characterized what we call *optimal risk sharing* (ORS) rules: POAs fulfilling the IR constraints. All these results hold independently of the fact that the involved criterions $(U_i)_{i=1}^n$ are or are not monotone. However, it is interesting to point out if the preferences of the agents do or do not satisfy this property, in order to compare the problem that we must solve with the one which only involves "ad hoc" monotone functionals. Therefore, for any functional U_i we consider its best monotone-adjusted version, based on Maccheroni et al. [59], and study the new problem that arises. Besides these general results, we also study particular sup-convolution problems which involve well-known choice functionals and provide the explicit calculation of their optimal solutions. These examples also reveal peculiar attitudes of the agents characterized by such preference criterions: the conservative behaviour of the entropic-agent, as well as the inclination of the $AV@R$-agent towards taking extreme risks.

Another problem considered in this thesis is the first one mentioned above, that is, the problem of an agent who trades in a financial market so as to maximize the expected utility of her terminal wealth. We study it in Part II (Chapters 6-8), where we work in a security market consisting in d risky assets whose discounted prices are described by a

$\mathbb{R}^d-$valued locally-bounded semimartingale process S. We allow the wealth to be negative, hence taking utility functions defined on the entire real line. We assume the same context as Schachermayer [69], which ensures the existence and uniqueness of the solution to this optimization problem (that we call the *primal problem*). As many authors do, we use the powerful tool of convex duality theory in order to formulate the associated *dual problem* that, in this case, is expressed in terms of local-martingale measures and admits a unique solution as well (see [6]). Proceeding in this way, two mutually exclusive situations are singled out: the optimal martingale measure $\widehat{\mathbb{Q}}$ (unique solution to the dual problem) is equivalent to the objective probability measure $\mathbb{P}$, or $\widehat{\mathbb{Q}}$ is just absolutely continuous with respect to $\mathbb{P}$. Thus we say that we are in the *equivalent case* or in the *absolutely-continuous case*, respectively. In the former, the optimal terminal wealth $\widehat{X}$ (unique solution to the primal problem) is shown to be perfectly replicable, that is, achievable by optimally investing in the market (see [69, Theorem 2.2]). Therefore, $\widehat{X}$ can be represented as the final value of a stochastic integral with respect to the price process S. Otherwise, if $\widehat{\mathbb{Q}}$ is only absolutely continuous with respect to $\mathbb{P}$, there exists no self-financing trading strategy which perfectly replicates the optimal wealth, hence we lose the integral representation. Whereas some authors assume the equivalent case, either directly or by giving sufficient conditions to ensure its occurrence (see, e.g., [41], [34], [23] and, in particular, [69] when showing the integral representability), we go in the opposite direction. Our results hold true in both cases and become significant in the absolutely-continuous one. Therefore we often emphasize this anomalous case, where the optimal measure $\widehat{\mathbb{Q}}$ is not equivalent to $\mathbb{P}$, and the optimal wealth $\widehat{X}$ is infinite with strictly positive probability. In order to approximate the solution to the original problem, we introduce a sequence of optimization problems and prove that they fit in with the equivalent case. The solutions of these auxiliary problems are shown to be convergent to the solution of the original one, so that $\widehat{X}$ results to be attainable as the limit of terminal values for some self-financing trading strategies. In particular this means that, by trading in the market, we can achieve a wealth that is as big as desired in a set of positive probability. We also show that the absolutely-continuous case may occur for any utility function fulfilling our requests, so that our study is not in vain. For all the results in Part II, compare the recent paper of the author [1].

* * * * *

The thesis is organized as follows.

In Chapter 2 we fix our notations and recall some results of convex analysis that we strongly use in the thesis. We introduce the concept of utility as a tool that allows us to represent agents' preferences, emphasizing the fact that this can be expressed in several ways. A brief discussion on some characterizing properties of choice functionals concludes this introductive part.

Part I (§ 3-5) is dedicated to the problem of optimal risk sharing, and the approach we adopt is the same as Barrieu and El Karoui [4, 5], Jouini et al. [49]. In particular:

In Chapter 3 we define and study the family of functionals that we admit as agents' choice criterions. To start with we introduce the class of *monetary utility functionals*, for which we recall some known results and, especially, their representability in terms of probability measures under the Fatou property. The cases of the Average Value at Risk, the entropic and the semi-deviation utilities, are specifically studied with particular regard to their dual formulation. Afterwards, we relax the requirements made on the choice functionals, allowing non-monotone preferences as well. We extend some of the results given for monetary utility functionals to this wider class of criterions, and we especially prove that the representation in terms of $\sigma-$additive measures holds under the Fatou property. In particular we present and accurately study the two most famous and useful non-monotone choice criterions: the mean-variance and the standard-deviation principles. Chapter 3 ends with the introduction of the *best* monotone approximation of non-monotone preference functionals.

In Chapter 4 we formulate our optimization problem, which is a constrained sup-convolution problem. Existence results for the solutions to this problem are given under the assumption of law-invariance for all the involved choice functionals. Once this is done, we study optimal risk sharing problems involving the best monotonicity-adjusted versions of the non-monotone preference criterions.

In Chapter 5 we consider and explicitly solve problems of optimal sharing of aggregate risks among agents endowed with well known preference relations. We also characterize the solutions of problems involving agents with particular attitudes, such as *strict risk-aversion conditionally on any event*.

Part II (§ 6-8) deals with the expected utility maximization problem, hence the characterization of the optimal investment process. In particular:

In Chapter 6 we formalize the problem: given an agent whose preferences are represented by a utility function u defined on $\mathbb{R}$, we aim at maximizing the expected utility of the wealth she can achieve, at time horizon T, by trading in the market. By means of the dual formulation of this problem, which consists in minimizing the generalized entropy among all absolutely continuous local-martingale measures, we characterize the two cases: "equivalent" and "absolutely-continuous".

In Chapter 7 we focus our attention on the absolutely-continuous case, where the optimal martingale measure is not equivalent to the historical probability measure. In this setting we introduce and study a sequence of optimization problems, defined on some random trading intervals contained in $[0, T]$, whose solutions converge to the solution of the original problem.

In Chapter 8 we conclude the thesis with the construction of a security market model where the martingale measure minimizing the generalized entropy is not equivalent to the historical probability, hence our analysis becomes significant.

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Chapter 2

Preliminaries

2.1 Set Up and Notations

Consider two dates: today $t = 0$, where everything is known, and a fixed future date $T \in (0, +\infty]$, where a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given. Here the set Ω describes all possible scenarios, the σ-algebra $\mathcal{F}$ models our knowledge at time T, and the probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is the so-called historical (or objective) probability. If no trading is possible between 0 and T, we are in a stating situation and, precisely, in the simple model consisting of these two dates. This is, for example, the situation considered in Chapters 3-5, where we study the problem of optimal sharing of risk among several economic agents. On the other hand, if trading is possible at any time t in a given time index set $\mathcal{T} \subseteq [0, T)$, $0 \in \mathcal{T}$, then we are in a dynamic setting and we need to supply the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$. In this case we consider $\mathcal{F}_T = \mathcal{F}$, whereas $\mathcal{F}_t$ can be seen as the information available at time t, for any $t \in \mathcal{T}$. As usual, the filtration is assumed to satisfy the conditions of saturatedness and right continuity. This is the situation of Chapters 6-8, where we discuss the problem of maximizing the expected utility in a continuous-time market model, thus allowing trading at any date between today and the time horizon T: $\mathcal{T} = [0, T)$.

We write $L^0 := L^0(\Omega, \mathcal{F}, \mathbb{P})$ for the class of all $\mathcal{F}$-measurable random variables, and $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$, with $1 \leq p < \infty$, for the family of all elements in L^0 with finite p-norm $\|.\|_{L^p} = (\mathbf{E}[|.|^p])^{1/p}$ (we use $\|.\|$ instead of $\|.\|_{L^1}$ when it does not generate confusion).

$L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is the space of all essentially bounded random variables, that is, all elements in L^0 with finite L^∞-norm $\|f\|_{L^\infty} = \operatorname{ess\,sup}_\omega |f(\omega)|$. With $(L^\infty)^* := L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \equiv \mathrm{ba}(\Omega, \mathcal{F}, \mathbb{P})$ we mean its topological dual, i.e., the set of all bounded, finitely additive measures μ with the property that $\mathbb{P}(A) = 0$ implies $\mu(A) = 0$. Notice that the inclusion $L^1 \subseteq (L^\infty)^*$ holds, by identification of σ-additive measures with their Radon-Nikodym derivatives.

We denote $\mathcal{P}$ as the collection of all probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to $\mathbb{P}$, and $\mathcal{Z}$ as the set of their Radon-Nikodym derivatives with respect to $\mathbb{P}$:

$$\mathcal{P} := \{\mathbb{Q} : \mathbb{Q}\,\text{probability measure},\ \mathbb{Q} \ll \mathbb{P}\}, \quad \mathcal{Z} := \{Z \in L^1_+ : \mathbf{E}[Z] = 1\}. \tag{2.1}$$

Analogously, we call $\mathcal{P}_\sigma$ the collection of all σ-additive measures absolutely continuous with respect to $\mathbb{P}$ and normalized to 1, and $\mathcal{Z}_\sigma$ the relative set in L^1:

$$\mathcal{P}_\sigma := \{\mu : \mu\ \sigma\text{-additive measure},\ \mu \ll \mathbb{P},\ \mu(\Omega) = 1\}, \quad \mathcal{Z}_\sigma := \{Z \in L^1 : \mathbf{E}[Z] = 1\}. \tag{2.2}$$

2.2 Some Necessary Functional Analysis

In this section we briefly recall some classical results of convex duality and differential theory (see, e.g., [28], [10], [67]) which we will frequently use of throughout the thesis. For the sake of simplicity, we introduce them in a version which may not be the most general, but certainly the most suitable for our later use.

Definition 2.1. *Let E be a topological vector space (TVS). A function $f : E \to [-\infty, +\infty]$ is said lower semi-continuous (l.s.c.) if*

$$\liminf_{x \to x_0} f(x) \geq f(x_0),\ \forall x_0 \in E,$$

whereas it is said upper semi-continuous (u.s.c.) if

$$\limsup_{x \to x_0} f(x) \leq f(x_0),\ \forall x_0 \in E.$$

Moreover, we say that f is *proper* if $\mathrm{dom}(f) := \{x \in E : f(x) \in \mathbb{R}\} \neq \emptyset$.

From now on, consider a real Banach space $(E, \|.\|_E)$ and its topological dual

$$E^* := \{f : E \to \mathbb{R} : f\,\text{continuous and linear}\}.$$

Definition 2.2. *Let* $\varphi : E \to [-\infty, +\infty[$ *be a proper concave function. Define* $\varphi^* : E^* \to]-\infty, +\infty]$*, the Fenchel-Legendre transform of* φ*, as*

$$\varphi^*(\mu) := \sup_{x \in E}\{\varphi(x) - \langle \mu, x\rangle\}, \ \forall \mu \in E^*. \tag{2.3}$$

Note that φ^* is a proper, convex and $\sigma(E^*, E)-$l.s.c. function on E^*, also called the *convex conjugate* of φ. Moreover, we can define the concave function $\varphi^{**} : E \to [-\infty, +\infty[$, *conjugate* of φ^*, as

$$\varphi^{**}(x) := \inf_{\mu \in E^*}\{\varphi^*(\mu) + \langle \mu, x\rangle\}, \ \forall x \in E. \tag{2.4}$$

It follows that φ^{**} is $\sigma(E, E^*)-$u.s.c. and $\varphi^{**} \geq \varphi$.

Theorem 2.3 (Fenchel-Moreau)**.** *If* φ *is proper, concave and* $\sigma(E, E^*)-$*u.s.c., then* $\varphi^{**} = \varphi$*. In this case we say that* φ *and* φ^* *are* $\langle E, E^*\rangle$*-conjugate.*

We can immediately see that

$$\langle \mu, x\rangle \geq \varphi(x) - \varphi^*(\mu), \ \forall x \in E, \ \forall \mu \in E^*, \tag{2.5}$$

i.e., the so-called *Fenchel's inequality* holds for any proper concave function φ and its conjugate φ^*.

Let $f : E \to [-\infty, +\infty[$ be a concave function. The superdifferential (or simply *differential*, when it does not generate confusion) of f at $x \in E$, is defined as follows

$$\partial f(x) = \{t \in E^* : f(y) \leq f(x) + \langle t, y - x\rangle, \forall y \in E\}, \tag{2.6}$$

and each element in $\partial f(x)$ is said a *supergradient* of f at x. Let $f : E \to]-\infty, +\infty]$ now be a convex function. The subdifferential (or *differential*) of f at $x \in E$, is

$$\partial f(x) = \{t \in E^* : f(y) \geq f(x) + \langle t, y - x\rangle, \forall y \in E\}, \tag{2.7}$$

and each element in this set is called a *subgradient* of f at x.

Proposition 2.4. *Let* f *be a proper, concave (resp. convex) and u.s.c. (resp. l.s.c.) function on* E*. Then* f *is also* $\sigma(E, E^*)-$*u.s.c. (resp.* $\sigma(E, E^*)-$*l.s.c.).*

As consequence of this proposition, we have that the Fenchel-Moreau theorem is still true if we replace the upper semi-continuity w.r.t. the weak topology with the upper semi-continuity w.r.t. the strong topology, easier to check.

The following theorem will ensure the existence of solutions to some optimization problems considered in Part I of this thesis.

Theorem 2.5. *Let f be a proper, concave (convex) and continuous function on E. Then, for every $x \in E$, $\partial f(x) \neq \emptyset$.*

Consider φ and φ^* as in Definition 2.2 and fulfilling the Fenchel-Moreau theorem. We point out that the graphs of the multifunctions

$$\partial\varphi(x) = \{\mu \in E^* : \varphi(y) \leq \varphi(x) + \langle \mu, y - x\rangle, \, \forall y \in E\} \tag{2.8}$$

and

$$\partial\varphi^*(\mu) = \{x \in E : \varphi^*(\nu) \geq \varphi^*(\mu) + \langle \nu - \mu, x\rangle, \, \forall \nu \in E^*\}, \tag{2.9}$$

characterize the pairs (x, μ) and $(-x, \mu)$ respectively, such that Fenchel's inequality (2.5) is actually an equality for (x, μ). In fact, we have a stronger result stated in the following theorem.

Theorem 2.6. *Consider φ and φ^* as above. Then, for any $x \in E$ and $\mu \in E^*$,*

$$\mu \in \partial\varphi(x) \iff x \in -\partial\varphi^*(\mu) \iff \varphi(x) = \varphi^*(\mu) + \langle \mu, x\rangle. \tag{2.10}$$

In particular, whenever $\partial\varphi(x) \neq \emptyset$, then

$$\partial\varphi(x) = \arg\min{}_\mu\{\varphi^*(\mu) + \langle \mu, x\rangle\},$$

and whenever $\partial\varphi^*(\mu) \neq \emptyset$, then

$$\partial\varphi^*(\mu) = -\arg\max{}_x\{\varphi(x) - \langle \mu, x\rangle\}.$$

Theorem 2.7. *Let φ be a proper concave function. For $x \notin dom(\varphi)$, $\partial\varphi(x)$ is empty. For $x \in ri(dom(\varphi))$, $\partial\varphi(x)$ is not empty and $\varphi(x) = \varphi^{**}(x)$.*

The following notion of differentiability will permit us to give a necessary and sufficient condition to reduce the differential of a function to a singleton. Consider a function $f : E \to [-\infty, +\infty]$. If the following limit

$$J_f(x; y) \quad := \quad \lim_{\epsilon \to 0} \frac{f(x + \epsilon y) - f(x)}{\epsilon} \tag{2.11}$$

exists, we call it the Gateaux differential of f at x with perturbation y. Moreover, if (2.11) define a linear functional $J_f(x; .)$ on E, then f is said Gateaux differentiable (or just *differentiable*) and we denote by $\nabla f(x) := J_f(x; .)$ the Gateaux differential of f at x.

Theorem 2.8. *Let $f : E \to [-\infty, +\infty]$ be a concave (resp. convex) proper function, and let $x \in dom(f)$. Then*

(i) if f is differentiable at x, then $\nabla f(x)$ is the unique supergradient (reps. subgradient) of f at x: $\partial f(x) = \{\nabla f(x)\}$,

(ii) if f has a unique supergradient (reps. subgradient) at x, then f is differentiable at x.

Theorem 2.9. *An u.s.c. (resp. l.s.c.) proper concave (resp. convex) function is essentially strictly concave (resp. convex) if and only if its conjugate is essentially differentiable.*

Proposition 2.10. *Let $B \in L^\infty$ be a convex set. Then B is $\sigma(L^\infty, L^1)-$closed iff for each $n \in \mathbb{N}$, the set $B_n := \{X : X \in B, \|X\|_{L^\infty} \leq n\}$ is closed w.r.t. the convergence in probability.*

2.3 Functionals Modelling Preferences

Financial positions in the future date T are described by elements in $L^0(\Omega, \mathcal{F}, \mathbb{P})$. Now we want to evaluate them by means of functionals fulfilling suitable properties. In order to do this, we fix a set $\mathcal{X} \in L^0$ as the collection of all admissible payoff profiles at time T. The preferences of an economic agent can be expressed by means of a binary relation " $\lesssim$ " on $\mathcal{X}$, satisfying the two following properties:

(i) completeness: for all $X, Y \in \mathcal{X}$, either $X \lesssim Y$ or $Y \lesssim X$ or both are true,

(ii) transitivity: if $X \lesssim Y$ and $Y \lesssim Z$, then also $X \lesssim Z$.

At this point, we say that a functional $U : \mathcal{X} \to \mathbb{R}$ represents a preference relation " $\lesssim$ ", if

$$U(X) \leq U(Y) \qquad \Longleftrightarrow \qquad X \lesssim Y. \tag{2.12}$$

We call it *choice functional* or *preference functional*, and consider the value it associates to a financial position as an indicator of the level of satisfaction/riskiness derived from this position. More precisely, relating to (2.12), we indistinctly say that payoff Y is preferred to payoff X, or that X is riskier than Y. Different preference relations clearly lead to different choice functionals with different characteristics. The appropriate one depends of course on the economics of the situations it is used for.

As already mentioned in the Introduction, a classical way to evaluate financial positions is to consider the expected utility. In this case the preferences of an investor are expressed as follows:

$$U(X) := \mathbf{E}[u(X)], \quad \forall\, X \in \mathcal{X}, \tag{2.13}$$

for some *utility function* u defined on an interval $I \subseteq \mathbb{R}$ (usually $\mathbb{R}$ or $\mathbb{R}^+$), which is strictly concave, strictly increasing and continuous on I. This is exactly our point of view in Chapters 6-8, where we discuss the problem of constructing a portfolio which maximizes, under a given budget constraint, the expected utility of the resulting payoff.

On the other hand, by means of the expected utility, we can also introduce another interesting type of choice functionals. Consider a utility function u on $\mathbb{R}$ and define the family of positions with expected utility bounded from below by the threshold $u(x)$, for some $x \in \mathbb{R}$:

$$\mathcal{A}(u) := \{X \in \mathcal{X} : \mathbf{E}[u(X)] \geq u(x)\}. \tag{2.14}$$

To regard the convex set $\mathcal{A}(u)$ as the class of acceptable financial positions from the point of view of a supervising agency, leads to the following definition:

$$U(X) := \sup\{m \in \mathbb{R} : X - m \in \mathcal{A}(u)\}, \tag{2.15}$$

which characterizes a choice functional satisfying desirable properties, so that it belongs to the remarkable class of monetary utility measures (see §3).

However, as well argued in [40], there is no set of properties which is suitable for all types of risky situations: according to the case considered, there will be characteristics that are preferable to other ones. Therefore at this point we are deliberately vague on the properties we will require for preference functionals and only a brief comment will be given on some of these. We refer to [3], [33] and [36] for a broader discussion.

Fix $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and consider the following axioms for a proper functional $U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$:

(ph) positive homogeneity: $U(\alpha X) = \alpha U(X),\ \forall\, X \in L^\infty, \alpha \geq 0$;

(c) concavity: $U(\alpha X + (1-\alpha)Y) \geq \alpha U(X) + (1-\alpha)U(Y),\ \forall\, X, Y \in L^\infty, \alpha \in (0,1)$;

(sl) super-linearity: $U(X+Y) \geq U(X)+U(Y)$, $\forall\, X, Y \in L^\infty$;

(ci) cash-invariance: $U(X+c) = U(X)+c$, $\forall\, X \in L^\infty, c \in \mathbb{R}$;

(n) normalization: $U(0) = 0$;

(m) monotonicity: $U(X) \geq U(Y)$, $\forall\, X, Y \in L^\infty$ s.t. $X \geq Y$;

(li) law-invariance: $U(X) = U(Y)$, $\forall\, X, Y \in L^\infty$ with the same distribution.

Positive homogeneity makes U linear on proportional payoffs, ignoring the problem of liquidity and, therefore, the fact that the risk often increases more than linearly with the losses. Under this property we have the equivalence of the other two: concavity, which expresses the incentive to the diversification of the risk; and super-linearity, which means that a merger cannot create extra risk. This last axiom indicates that a position can decrease the riskiness of another one, which may or may not be desirable. For example, if two payoffs "go in the same direction", there shouldn't be any reason to assume super-additivity, as well argued in Section 3.4 with regard to the mean-variance principle. In particular, we may want the opposite situation to occur, i.e. sub-linearity, when dealing with affiliated companies, in order to allow the reduction of the risk when sharing it among them.

The axiom of cash-invariance (or translation-invariance) is new with respect to the classical theory of risk. For example (*ci*) is not satisfied by the expected utility, but we can get it back by considering a functional of type (2.15) instead of (2.13). Translation-invariance means that the increment of U obtained by adding a deterministic quantity, just equals this quantity, and this allows us to consider $U(X)$ as expressed in monetary terms (in particular, in the same unit in which X is expressed). This characteristic fully conforms with the interpretation of $-U(X)$ as the capital requirement needed to make X acceptable from the point of view of a supervising agency, and this leads to the characterization of $U(X)$ in terms of a certain set of acceptable future positions (the so-called *acceptance set*). If $U(X)$ is negative, then it is the minimal extra cash to be added to the prospect X to make it acceptable. If, on the other hand, it is positive, it represents the maximal amount which the investor can withdraw from her portfolio X without changing the acceptability of this position. In Chapters 3-5 we assume this property satisfied, and this allows us to consider, without loss of generality, that the condition of normalization also holds true, by possibly adding a constant to U.

This request for invariance also favours another type of discussion regarding the desirability of property (ph), often confused with currency independence. Obviously the required capital should not depend on the currency in which payoffs and choice functionals are expressed. However, this independence has nothing to do with positive homogeneity, as can be seen by the argument presented in Section 3.4. For a further understanding of this topic, we refer to a discussion presented in Goovaerts et al. [40].

With regard to the monotonicity, it seems to be the most natural requirement for a functional associated to a preference relation. Indeed it appears reasonable to assume that any rational individual prefers "more" to "less". Nevertheless, in the problem of optimal risk sharing, we drop this assumption in order to study a wider class of criterions.

Lastly, invariance with respect to distribution is a fundamental property used in our study. To require that a choice criterion satisfies (li), means that the degree of satisfaction/riskiness of a future financial position does not depend on where the payoff takes its values, but only on its law under $\mathbb{P}$. The importance of this axiom derives from the fact that it leads to an easier dual representability (see §3) and ensures the existence of solutions to particular optimization problems (see §4).

Part I

Optimal Risk Sharing with Non-Monotone Choice Criterions

Chapter 3

Representation of Choice Functionals

In this chapter (as in §4 and §5) we consider the situation where the set $\mathcal{X}$ of possible financial positions occurring in T, consists of all essentially bounded measurable functions on $(\Omega, \mathcal{F}, \mathbb{P})$. Therefore, we deal with choice functionals defined on $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ which fulfil suitable properties. Following the axiomatic approach of Artzner et al. [3], we first introduce the class of monetary utility functionals (Definition 3.1) and recall some fundamental results, capturing their properties. We then enlarge this class to the family of choice criterions characterized by Assumption 3.20. This fact permits us to include in our study agents endowed with non-monotone preference relations and to generalize some known results.

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is assumed to be standard, which in particular means that it is non-atomic, i.e., it is support of continuous random variables.

3.1 Monotone Choice Functionals

Let us call to mind the axioms introduced in the previous chapter.

Definition 3.1. *A proper functional $U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, with $U(0) = 0$, is called monetary utility measure or monetary utility functional (m.u.f.) if it is concave, monotone and cash-invariant.*

Note that, by switching from a m.u.f. U to the associated loss functional $\rho := -U$, we get back the so-called *convex risk measures*, in the sense of Föllmer-Schied [33]. From this it is clear that, by imposing on U the additional axiom of positive homogeneity, we obtain a *coherent utility measure* (with the meaning that the corresponding loss functional is a coherent risk measure, as introduced in [3]). We make the choice to deal with utility measures, instead of risk measures, in order to immerse our study in the classical utility theory.

Remark 3.2. *The axioms of monotonicity and cash-invariance ensure that U is Lipschitz-continuous with respect to the supremum norm on L^∞.*

This regularity leads to the application of important results of functional analysis. In particular, given $U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ monetary utility functional and $V : L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ its Fenchel-Legendre transform:

$$V(\mu) = \sup_{X \in L^\infty} \{U(X) - \langle \mu, X \rangle\}, \ \forall \mu \in (L^\infty)^*, \tag{3.1}$$

the Fenchel-Moreau theorem ensures that U and V are $\langle L^\infty, (L^\infty)^* \rangle$-conjugate. As a consequence U can be represented in the following way:

$$U(X) = \inf_{\mu \in (L^\infty)^*} \{V(\mu) + \langle \mu, X \rangle\}, \ \forall X \in L^\infty, \tag{3.2}$$

and we can study its properties by means of the convex dual function V. In literature V is often called *minimal penalty function* of U and clearly satisfies:

(i) $V(\mu) = +\infty$, for any μ such that $\mu(1) \neq 1$,

(ii) $V(\mu) = +\infty$, for any μ not positive,

where we say that a measure μ is "positive" if it produces $\langle \mu, Y \rangle \geq 0$ whenever $Y \geq 0$. Assertion (i) follows by cash-invariance property, since

$$\begin{aligned} V(\mu) &= \sup_{X \in L^\infty} \{U(X) - \langle \mu, X \rangle\} \geq \sup_{c \in \mathbb{R}} \{U(c) - \langle \mu, c \rangle\} \\ &= \sup_{c \in \mathbb{R}} \{c(1 - \mu(1))\} = +\infty, \end{aligned}$$

for any $\mu \in (L^\infty)^*$ with total mass different from 1. On the other hand, monotonicity of U leads to (ii): let $Y \in L^\infty$ be a non-negative random variable, and $\mu \in (L^\infty)^*$ be such that

$\langle \mu, Y \rangle < 0$, then

$$\begin{aligned} V(\mu) &= \sup_{X \in L^\infty} \{U(X) - \langle \mu, X \rangle\} \geq \sup_{c \in \mathbb{R}^+} \{U(cY) - \langle \mu, cY \rangle\} \\ &= \sup_{c \in \mathbb{R}^+} \{U(cY) - c\langle \mu, Y \rangle\} = +\infty, \end{aligned}$$

as previously declared.

3.1.1 Some Representation Results

The representation theory of convex risk measures on L^∞ was developed by Delbaen [21, 20] and successively extended by Föllmer and Schied [32, 33], Frittelli and Rosazza Gianin [35, 36], among others. Here we present some results which we later use and extend upon. In particular, the following theorem provides a characterization of a particular class of monetary utility functionals, for which the dual representation in (3.2) is still true if we minimize over L^1 instead of the whole dual space $(L^\infty)^*$. This means that U can be described in terms of probability measures, as indicated in (3.3), where linear functionals take the form of expectations under these measures.

Theorem 3.3 ([20, 32]). *Let $U : L^\infty \to \mathbb{R}$ be a monetary utility functional and $V : (L^\infty)^* \to [0, \infty]$ its convex conjugate. Then the following statements are equivalent:*

(i) U can be represented by V restricted to the set of measures $\mathcal{P}$ (defined in (2.1)):

$$U(X) = \inf_{\mathbb{Q} \in \mathcal{P}} \{V(\mathbb{Q}) + \boldsymbol{E}_{\mathbb{Q}}[X]\}, \ \forall X \in L^\infty, \tag{3.3}$$

(ii) U is continuous from above: if $X_n \searrow X$ $\mathbb{P}$-a.s. then $U(X_n) \searrow U(X)$,

(iii) U satisfies the Fatou property: for any bounded sequence $(X_n)_n$ converging $\mathbb{P}$-a.s. to some X, then $U(X) \geq \limsup_n U(X_n)$.

By identifying a probability measure $\mathbb{Q}$ with its Radon-Nikodym derivative $\dfrac{d\mathbb{Q}}{d\mathbb{P}}$, we can analogously write

$$U(X) = \inf_{Z \in \mathcal{Z}} \{V(Z) + \mathbf{E}[ZX]\}, \tag{3.4}$$

where $\mathcal{Z}$ is the set of densities defined in (2.1).

The dual formulation of U in L^1 clearly leads to more manageable situations than those arising when dealing with $(L^\infty)^*$, especially if one wants to explicitly solve particular optimization problems, as the case in Chapter 5. Therefore, it is important to recognize when duality works in L^1, as for example under the assumption of law-invariance (compare [48]). Moreover, for a m.u.f. satisfying the law-invariance property, we have an interesting characterization in terms of quantile functions (obtained by Kusuoka [58] and further extended in [37] and [48]), which we present in Theorem 3.5. It relies on Lemma 3.4, which also allows us to extend this characterization to the class of non-monotone functionals (see Theorem 3.23).

Let us first recall that, for any random variable $X : \Omega \to \mathbb{R}$, the (lower-)quantile function is defined as $q_X(\alpha) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}$, for all $\alpha \in (0, 1]$, where $F_X(x) = \mathbb{P}(X \leq x)$ is the cumulative distribution function associated to X.

Lemma 3.4 ([33])**.** *For $X \in L^\infty$ and $Y \in L^1$, we have*

$$\int_0^1 q_X(t) q_Y(t) dt = \sup_{\widetilde{X} =_d X} \boldsymbol{E}[\widetilde{X} Y], \tag{3.5}$$

where $\widetilde{X} =_d X$ means that $\widetilde{X}$ and X have the same distribution.

Theorem 3.5. *Let U be a law-invariant monetary utility functional and let $\mathcal{A}_U$ be the associated set of acceptable positions (the so-called acceptance set of U), given by*

$$\mathcal{A}_U := \{X \in L^\infty : U(X) \geq 0\}. \tag{3.6}$$

Then we have the following representation for U:

$$U(X) = \inf_{Z \in \mathcal{Z}} \Big\{ - \int_0^1 q_{-Z}(t) q_X(t) dt + V(Z) \Big\}, \ \forall X \in L^\infty, \tag{3.7}$$

and the following one for its dual transform V:

$$V(Z) = \sup_{X \in L^\infty} \Big\{ U(X) + \int_0^1 q_{-Z}(t) q_X(t) dt \Big\} = \sup_{X \in \mathcal{A}_U} \Big\{ \int_0^1 q_{-Z}(t) q_X(t) dt \Big\}, \ \forall Z \in \mathcal{Z}. \tag{3.8}$$

Here we have used the fact that law-invariant monetary utility functionals automatically satisfy the Fatou property, as shown in Jouini et al. [48].

We point out that the convex conjugates of positively homogeneous functions are indicator functions of convex sets, in the sense of the convex analysis. This means that, for any coherent utility measure (m.u.f. satisfying axiom (ph)), the associated penalty function V takes the form:

$$V(\mu) = \chi_{\mathcal{C}} \equiv \begin{cases} 0, & \text{if } \mu \in \mathcal{C}, \\ +\infty, & \text{otherwise}, \end{cases} \tag{3.9}$$

for a given convex set $\mathcal{C} \subseteq (L^\infty)^*$, which leads to the following dual representation for U:

$$U(X) = \inf_{\mu \in \mathcal{C}} \{\langle \mu, X \rangle\}, \ \forall X \in L^\infty. \tag{3.10}$$

We now introduce the concept of comonotonicity, referring to [26] for an overview of this topic. For two financial positions it expresses a special type of dependency (we can say that "they move in the same direction"), whereas for functionals defined on L^∞ it indicates linearity on such type of random variables (an assumption that for a m.u.f. is stronger then positive homogeneity, see [33]).

Definition 3.6. *Two random variables X and Y on $(\Omega, \mathcal{F}, \mathbb{P})$ are called comonotone if satisfying relation $(X(\omega_0) - X(\omega_1))(Y(\omega_0) - Y(\omega_1)) \geq 0$, $\mathbb{P} \otimes \mathbb{P}$-almost surely. A function $f : L^\infty \to \mathbb{R}$ is said comonotone if $f(X + Y) = f(X) + f(Y)$ for any pair (X, Y) of comonotone random variables.*

In addition to that, we say that X and Y are anticomonotone if $-X$ and Y are comonotone.

Lemma 3.7. *Let U be a law-invariant m.u.f. and let $Z \in \mathcal{Z}$ be in $\partial U(X)$ for some $X \in L^\infty$. Then X and Z are anticomonotone random variables.*

Proof. Since $\partial U(X) \neq \emptyset$, we can write

$$\partial U(X) = \arg\min_f \{V(f) + \mathbf{E}[fX]\}.$$

On the other hand, the dual conjugate V inherits the property of law-invariance and this leads to

$$\begin{aligned} U(X) &= \inf_{f \in \mathcal{Z}} \{V(f) + \mathbf{E}[fX]\} = \inf_{f \in \mathcal{Z}} \inf_{g \in \mathcal{Z}, g =_d f} \{V(f) + \mathbf{E}[gX]\} \\ &= \inf_{f \in \mathcal{Z}} \{V(f) + \inf_{g \in \mathcal{Z}, g =_d f} \mathbf{E}[gX]\}. \end{aligned}$$

Therefore, $Z \in \arg\min_f \{V(f) + \mathbf{E}[fX]\}$ is, among all the elements g in $\mathcal{Z}$ with $g =_d Z$, the one that minimizes $\mathbf{E}[gX]$. On the other hand, for fixed X and law of g, $\mathbf{E}[gX]$ becomes minimal for X and g anticomonotone (compare [33]), which completes the proof.

□

Consider the set $\mathcal{D}$ of all concave and non-decreasing functions $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$, which in particular contains elements of type $-\int_0^{\cdot} q_{-Z}(s)ds$ for any $Z \in \mathcal{Z}$.

Theorem 3.8 ([58])**.** *For a law-invariant monetary utility functional, the following conditions are equivalent:*

(i) U is comonotone,

(ii) there exists a unique $f_U \in \mathcal{D}$ such that for any $X \in L^\infty$ we have

$$U(X) = \int_0^1 q_X(t) f_U'(t)dt = \inf\left\{ \int_0^1 q_X(t) f'(t)dt \,:\, f \in \mathcal{D} \text{ and } f \leq f_U \right\}. \tag{3.11}$$

In particular, the concave function f_U characterizing U in the sense of (3.11), permits a simple description of the gradients of U, as stated in the following lemma, due to Jouini et al. [49] (which formulate it in a more general form).

Lemma 3.9. *Let U be a law-invariant comonotone m.u.f., characterized by the concave function $f_U \in \mathcal{D}$. Then, for any $X \in L^\infty$ and $Z \in \mathcal{Z}$, we have $Z \in \partial U(X)$ if and only if*

$$-\int_0^t q_{-Z}(s)ds \leq f_U(t) \quad \text{and} \quad q_X \quad \text{is constant on} \quad \left\{ -\int_0^t q_{-Z}(s)ds < f_U(t) \right\}. \tag{3.12}$$

3.1.2 Additional Properties

Here we recall two concepts given on functionals defined on L^∞ (introduced in [49] and [48] respectively). They allow us to characterize particular types of choice criterions and to solve optimization problems involving them.

Definition 3.10. *A choice functional U defined on L^∞ is said to be strictly risk-averse conditionally on any event if it satisfies the following property:*

(S) $U(X) < U(X\mathbf{1}_{A^c} + \boldsymbol{E}[X|A]\mathbf{1}_A)$ *for any* $A \in \mathcal{F}$ *and* $X \in L^\infty$ *with* $\mathbb{P}(A) > 0$ *and* $essinf_A X < esssup_A X$.

This means that, if an economic agent is characterized by such a choice functional, then she strictly prefers for averaging any event where the financial position is not a.s. constant. For functionals satisfying property (S) we can give a result analogous to that shown in [49] for law-invariant monetary utility functionals which are *strictly risk-averse conditionally on lower tail events.*

Lemma 3.11. *Let U be a concave functional on L^∞, strictly risk-averse conditionally on any event. For any pair $(X, Z) \in L^\infty \times L^1$ s.t. $Z \in \partial U(X)$, and any set $A \in \mathcal{F}$ s.t. $\mathbb{P}(A) > 0$ and Z constant on A, then X is a.s. constant on A as well.*

Proof. If such a pair $(X, Z) \in L^\infty \times L^1$ with $Z \in \partial U(X)$ and Z constant with positive probability does not exist, then there is nothing to prove. Otherwise, consider any pair (X, Z) satisfying these hypothesis, and assume that the thesis is not true, i.e., X is not constant on $A := \{Z = z\}$, for some $z \in \mathbb{R}$ such that $\mathbb{P}(A) > 0$. Since U satisfies property (S) and X is not constant on A, then

$$U(X) < U(\overline{X}) \quad \text{where} \quad \overline{X} := X\mathbf{1}_{A^c} + \mathbf{E}[X|A]\mathbf{1}_A. \tag{3.13}$$

On the other hand, by definition (2.6) of superdifferential and noting that $\mathbf{E}[Z\overline{X}] = \mathbf{E}[ZX]$, we obtain

$$U(\overline{X}) \leq U(X) + \mathbf{E}[Z(\overline{X} - X)] = U(X),$$

which is in contradiction to (3.13). □

Definition 3.12. *We say that a function f defined on L^∞ satisfies the Lebesgue property if for any bounded sequence $(X_n)_n$ converging $\mathbb{P}$-a.s. to some X, then $\lim_{n\to\infty} f(X_n) = f(X)$.*

This property is clearly stronger than the Fatou property, that is the continuity from above. Moreover, for a monetary utility functional U, it is shown to be just equivalent to the continuity from below (see [33]): for every sequence $(X_n)_n \in L^\infty$ increasing monotonically to $X \in L^\infty$, then $\lim_n U(X_n) = U(X)$.

Theorem 3.13 ([48])**.** *Let $U : L^\infty \to \mathbb{R}$ be a monetary utility functional and $V : (L^\infty)^* \to [0, +\infty]$ its convex conjugate. Then*

(i) if U satisfies the Lebesgue property, then $dom(V) = \{V < \infty\} \subseteq L^1$,

(ii) U satisfies (li) and the Lebesgue property if and only if V satisfies (li) and representation (3.4), and $\{Z \in \mathcal{Z} : V(Z) \leq c\}$ is uniformly integrable for all $c > 0$.

This means more than just representability in L^1: under the Lebesgue property the infimum in (3.3) is actually a minimum.

3.2 Examples of Monetary Utility Functionals

We separately study some useful monetary utility functionals, focusing our interest on the characterization of their convex conjugate functions and their differentials, since we need them to explicitly solve optimization problems (in §5). Note that any of these functionals is law-invariant and satisfies the Lebesgue property, which allows us to deal with the space $\mathcal{Z}$ of densities instead of the dual space $(L^\infty)^*$ and makes everything easier.

3.2.1 The Average Value at Risk

The most representative monetary utility functional among the comonotone ones, is the so-called *Average Value at Risk* (taken with the opposite sign) introduced in [3]:

$$U_\lambda := -AV@R_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda q_X(t)dt, \quad \lambda \in (0,1], \tag{3.14}$$

where the Value at Risk is defined as $V@R_t(X) = -q_X^+(t) = -\inf\{x \in \mathbb{R} : F_X(x) > t\}$ (we recall that $q_X(t) = q_X^+(t)$, with the exception of at most countably many t in $(0,1]$). Note that we cannot consider $-V@R_\lambda$ as choice criterion since it fails concavity, hence it does not yield convex optimization problems and leads to pathological situations in the optimal risk sharing problem.

On the other hand, the $AV@R$-criterion (3.14) recovers concavity property and allows several interesting interpretations. In particular, since we work in a non-atomic probability space, it satisfies

$$-AV@R_\lambda(X) = \inf\{\mathbf{E}[X|A] : \mathbb{P}(A) > \lambda\},$$

that is, it coincides with the so-called *worst conditional expectation*. If, moreover, we consider a payoff X having continuous distribution, we also obtain equivalence with the concept of *tail conditional expectation* (or *expected shortfall*):

$$-AV@R_\lambda(X) = \mathbf{E}[X|X \le q_X(\lambda)].$$

For any $\lambda \in (0,1]$, U_λ is a coherent utility measure and representation (3.10) holds with characterizing set $\mathcal{P}_\lambda$ consisting of those probability measures with density bounded by λ^{-1}:

$$\mathcal{P}_\lambda = \Big\{\mathbb{Q} \in \mathcal{P} : 0 \le \frac{d\mathbb{Q}}{d\mathbb{P}} \le \frac{1}{\lambda}\Big\}.$$

This means that its Fenchel-Legendre transform V_λ equals $\chi_{\mathcal{P}_\lambda}$, which leads to

$$-AV@R_\lambda(X) = \inf_{\mathbb{Q}\in\mathcal{P}_\lambda} \mathbf{E}_\mathbb{Q}[X], \quad \lambda \in (0,1]. \tag{3.15}$$

In line with this representation, we can also define

$$U_0 := -AV@R_0(X) = \inf_{\mathbb{Q}\in\mathcal{P}} \mathbf{E}_\mathbb{Q}[X],$$

which represents the *worst-case utility measure*, that is, the most conservative monetary utility measure.

Here we study the differential of U_λ in the interesting case $\lambda \in (0,1]$. Theorem 3.8 applies with $f_\lambda(t) := \frac{t}{\lambda} \wedge 1$ as the unique function in $\mathcal{D}$ such that

$$-AV@R_\lambda(X) = \int_0^1 q_X(t) f'_\lambda(t)dt, \quad \forall X \in L^\infty. \tag{3.16}$$

At this point Lemma 3.9 gives us the characterization of the gradients of U_λ: for any $X \in L^\infty$, $\partial U_\lambda(X)$ consists of such measures $\mathbb{Q} \in \mathcal{P}_\lambda$ for which $-\int_0^t q_{-Z}(s)ds \le f_\lambda(t)$, with $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$, and such that X is constant when this inequality holds strictly. Then a generic element Z in $\partial U_\lambda(X)$ can be written as

$$Z = \begin{cases} 1/\lambda, & \text{on } \{X < q_X(\lambda)\}, \\ \in [0, 1/\lambda], & \text{on } \{X = q_X(\lambda)\}, \\ 0, & \text{on } \{X > q_X(\lambda)\}, \end{cases}$$

such that $\mathbf{E}[Z] = 1$.

It is now easy, by means of Theorem 2.6, to capture the gradients of the dual function V_λ. Consider any $Z \in$dom$(V_\lambda) = \mathcal{P}_\lambda$. Roughly speaking, a position X in $\partial V_\lambda(Z)$ takes its biggest values where $Z = 1/\lambda$, the smallest ones where $Z = 0$, and it is constant on $\{Z \in (0, 1/\lambda)\}$.

3.2.2 The Entropic Utility

The functional $U_\gamma^{en} : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ given by

$$U_\gamma^{en}(X) := -\gamma \ln \mathbf{E}\big[\exp\big(-X/\gamma\big)\big], \quad \gamma > 0, \tag{3.17}$$

is called *entropic utility function*. It clearly satisfies all the axioms characterizing a monetary utility functional, and some of them are strictly verified. The strict monotonicity is shown in [49], where the property of strict risk-aversion conditionally on any event is also proved.

Here we show the strict concavity of U_γ^{en} on $L_0^\infty := \{X \in L^\infty : \mathbf{E}[X] = 0\}$. Without loss of generality we proceed with $\gamma = 1$. Fix $\alpha \in (0,1)$ and consider the function $f : \mathbb{R}_0^+ \to \mathbb{R}$ defined by

$$f(t) = 1 - \alpha + \alpha t - t^\alpha.$$

This function admits a unique minimum in $t = 1$, where $f(1) = 0$. Therefore, $f \geq 0$ on $\mathbb{R}_0^+$ and the equality holds iff $t = 1$. For any pair of random variables $X, Y \in L_0^\infty$, $X \not\equiv Y$, we define

$$t := \exp(-X + Y) \cdot \frac{\mathbf{E}[\exp(-Y)]}{\mathbf{E}[\exp(-X)]}.$$

Since $X \not\equiv Y$, we have that $\beta := \mathbb{P}(t \neq 1) > 0$. By the positivity of f, we get

$$\exp(-\alpha X + \alpha Y)\left(\frac{\mathbf{E}[\exp(-Y)]}{\mathbf{E}[\exp(-X)]}\right)^\alpha \leq \alpha \exp(-X + Y)\frac{\mathbf{E}[\exp(-Y)]}{\mathbf{E}[\exp(-X)]} + 1 - \alpha,$$

which we can rewrite as follows

$$\begin{aligned} \exp(-\alpha X - (1-\alpha)Y) \quad \leq \quad & \alpha \exp(-X)\left(\frac{\mathbf{E}[\exp(-Y)]}{\mathbf{E}[\exp(-X)]}\right)^{1-\alpha} + \\ & (1-\alpha)\exp(-Y)\left(\frac{\mathbf{E}[\exp(-Y)]}{\mathbf{E}[\exp(-X)]}\right)^{-\alpha}, \end{aligned}$$

where the strict inequality holds with probability $\beta > 0$. By taking the expectation of both sides we obtain

$$\mathbf{E}[\exp(-\alpha X - (1-\alpha)Y)] < (\mathbf{E}[\exp(-X)])^\alpha (\mathbf{E}[\exp(-Y)])^{1-\alpha}$$

and, by passing to the logarithm, we have the desired result:

$$\ln \mathbf{E}[\exp(-(\alpha X + (1-\alpha)Y))] < \alpha \ln \mathbf{E}[\exp(-X)] + (1-\alpha)\ln \mathbf{E}[\exp(-Y)].$$

Note that we can obtain the choice functional defined in (3.17) by relying on the classical exponential utility function $u_\gamma : \mathbb{R} \to \mathbb{R}^-$, given by

$$u_\gamma(x) = -\exp(-x/\gamma), \ \ \gamma > 0.$$

Indeed, if we define the set $\mathcal{A}(u_\gamma)$ as in (2.14):

$$\mathcal{A}(u_\gamma) := \{X \in L^\infty : \mathbf{E}[u_\gamma(X)] \geq -1\}$$

and the relative utility functional as in (2.15):

$$U_\gamma(X) := \sup\{x \in \mathbb{R} : X - x \in \mathcal{A}(u_\gamma)\},$$

then we get $U_\gamma^{en} = U_\gamma$ on L^∞, and $\mathcal{A}(u_\gamma)$ as the acceptance set (3.6) associated to U_γ^{en}.

Remark 3.14. *By dominated convergence, one can easily verify that the functional defined in (3.17) satisfies the Lebesgue property. Then, by Theorem 3.13, the effective domain of its convex conjugate V_γ^{en} is contained in L^1 and, in particular, in $\mathcal{Z}$, by the argument that follows (3.2).*

Recall that, for any probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, the relative entropy w.r.t. $\mathbb{P}$ is given by

$$H(\mathbb{Q};\mathbb{P}) = \begin{cases} \mathbf{E}\Big[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big], & \text{if } \mathbb{Q} \ll \mathbb{P}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.18}$$

In the following theorem we state a strict link between this function and the entropic utility function, which gives a justification for the name of the latter.

Proposition 3.15. *Let $H(.\,;\mathbb{P})$ be the relative entropy (3.18) and $\mathbb{Q}$ any measure in $\mathcal{P}$. Then we have the equality:*

$$H(\mathbb{Q};\mathbb{P}) = \sup_{X \in L^\infty} \{-\ln \boldsymbol{E}[\exp(-X)] - \boldsymbol{E}_{\mathbb{Q}}[X]\} \tag{3.19}$$

which, for any $\gamma > 0$ and $U_\gamma^{en}(X)$ defined in (3.17), can be rewritten as

$$\gamma H(\mathbb{Q};\mathbb{P}) = \sup_{X \in L^\infty} \{U_\gamma^{en}(X) - \boldsymbol{E}_{\mathbb{Q}}[X]\}. \tag{3.20}$$

From Remark 3.14 and relation (3.1), we have that (3.20) completely describes the dual conjugate V_γ^{en} of the entropic utility. In fact, we can draw it out from the following theorem which provides a full description of the duality between U_γ^{en} and V_γ^{en}.

Theorem 3.16. *Let $U_\gamma^{en} : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be the m.u.f. defined in (3.17) and $V_\gamma^{en} : L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ its convex conjugate function. Then*

(i) $V_\gamma^{en}(Z) = \gamma H(\mathbb{Q}; \mathbb{P})$, where $Z = \dfrac{d\mathbb{Q}}{d\mathbb{P}}$, for all $Z \in \mathcal{Z}$,

(ii) $\partial U_\gamma^{en}(X) = \left\{ \dfrac{\exp(-X/\gamma)}{\|\exp(-X/\gamma)\|} \right\}$, for all $X \in L^\infty$,

(iii) $\partial V_\gamma^{en}(Z) = \{\gamma \ln Z + c, \forall c \in \mathbb{R}\}$, for all $Z \in$ dom(V^{en}).

Therefore, we can write the dual representation formula for U:

$$U_\gamma^{en}(X) = \inf_{\mathbb{Q} \in \mathcal{P}} \left\{ \gamma \mathbf{E}_\mathbb{Q} \left[\ln \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] + \mathbf{E}_\mathbb{Q}[X] \right\}. \tag{3.21}$$

Proof. From Remark 3.14, we know that the duality works in L^1. Moreover, strict concavity of U_γ^{en} on L_0^∞ implies, by Theorem 2.9, that the differential of V_γ^{en} at any Z in its domain is a singleton in L_0^∞. This means that $\partial V_\gamma^{en}(Z)$ consists of a unique element in L^∞ up to an additive constant, and the recipe to find it, is

$$\partial V_\gamma^{en}(Z) = -\arg\max_{X \in L^\infty} \{U_\gamma^{en}(X) - \mathbf{E}[ZX]\}.$$

For $Z \in \mathcal{Z}$, the functional f on L^∞ given by

$$f(X) := U_\gamma^{en}(X) - \mathbf{E}[ZX]$$

is well defined, concave and Gateaux differentiable, with differential

$$\nabla f(X) = -Z + \frac{\exp(-X/\gamma)}{\|\exp(-X/\gamma)\|}.$$

Noticing that $X_Z := -\gamma \ln Z$ solves $\nabla f(X_Z) = 0$ and that f attains its maximum at X_Z, we obtain (iii). Now, by Theorem 2.6 and $\mathbf{E}[Z] = 1$, (ii) immediately follows and we are able to compute the dual transform

$$V_\gamma^{en}(Z) = U_\gamma^{en}(X_Z) - \mathbf{E}[ZX_Z] = \gamma \mathbf{E}[Z \ln Z], \ \forall Z \in \mathrm{dom}(V_\gamma^{en}),$$

as stated in (i). □

Proof. [Proposition 3.15] It directly follows from Theorem 3.16, restating the duality relation between U_γ^{en} and V_γ^{en}. □

3.2.3 The Semi-Deviation Utility

Consider, for any $1 \le p \le \infty$, the functional $U^p_\delta : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ defined as follows:

$$U^p_\delta(X) := \mathbf{E}[X] - \delta \|(X - \mathbf{E}[X])^-\|_{L^p}\,,\ \ 0 < \delta \le 1. \tag{3.22}$$

It is called *semi-deviation utility* and, for any $1 \le p \le \infty$, it is a positively homogeneous monetary utility functional (see, e.g., [29]), comonotone if and only if $p = \infty$. Therefore, its Fenchel-Legendre transform V^p_δ is the indicator function of some convex set $\mathcal{C}^p \subseteq \mathcal{Z}$, in the sense of the convex analysis:

$$V^p_\delta(Z) = \chi_{\mathcal{C}^p}\,, \tag{3.23}$$

which leads to the following representation:

$$U^p_\delta(X) = \inf_{Z \in \mathcal{C}^p} \mathbf{E}[ZX].$$

In particular we are interested in the case $1 < p < \infty$, where U^p_δ exhibits properties such as strict monotonicity and strict risk-aversion conditionally on lower tail events (see [49]): for any $X \in L^\infty$ and $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ and $\mathrm{essinf}_A X < \mathrm{esssup}_A X \le \mathrm{essinf}_{A^c} X$, then $U^p_\delta(X) < U^p_\delta(X\mathbf{1}_{A^c} + \mathbf{E}[X|A]\mathbf{1}_A)$. Moreover it satisfies the Lebesgue property, as we can see by just applying the dominated convergence theorem, and therefore $\mathcal{C}^p$ is a subset of $\mathcal{Z}$, by Theorem 3.13. In fact, as shown in an example in [21], U^p_δ can be obtained by the set of probability measures with density in

$$\big\{1 + \delta(g - \mathbf{E}[g]) :\ g \ge 0, \|g\|_{L^q} \le 1\big\}, \tag{3.24}$$

where $q = p/(p-1)$ is the conjugate of p. Here we especially consider the case $p = 2$, for which we characterize the gradients of U^2_δ and V^2_δ as follows:

Theorem 3.17. *Let $U^2_\delta : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be the semi-deviation utility with parameters $p = 2$ and $\delta \in (0, 1]$, and let $V^2_\delta : L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ be its convex conjugate function. Define*

$$h(X) := \frac{(X - \boldsymbol{E}[X])^- - \|(X - \boldsymbol{E}[X])^-\|_{L^1}}{\|(X - \boldsymbol{E}[X])^-\|_{L^2}}\,, \quad \forall\, X \in L^\infty \setminus \{c :\ c \in \mathbb{R}\}.$$

Then we have:

(i) for any $X \in L^\infty$,

$$\partial U_\delta^2(X) = \begin{cases} dom(V_\delta^2), & \text{if } X = const, \\ \{1 + \delta h(X)\}, & otherwise, \end{cases}$$

(ii) for any $Z \in dom(V_\delta^2)$,

$$\partial V_\delta^2(Z) = \{c : c \in \mathbb{R}\} \cup \{X \in L^\infty : Z = 1 + \delta h(-X)\}. \tag{3.25}$$

In particular, $\big\{1 + \delta h(X) : \ X \in L^\infty \setminus \{c : c \in \mathbb{R}\}\big\}$ is the minimal set $C \subseteq \mathcal{Z}$ which allows us to represent U_δ^2 in the following form:

$$U_\delta^2(X) = \min_{Z \in C} \mathbf{E}[ZX], \ \forall X \in L^\infty.$$

Proof. For any $Z \in \mathcal{Z}$ we know that

$$\begin{aligned} V_\delta^2(Z) &= \sup_{X \in L^\infty} \{\mathbf{E}[X(1-Z)] - \delta \|(X - \mathbf{E}[X])^-\|_{L^2}\} && (3.26) \\ &= 0 \vee \sup_{X \in L^\infty, X \neq \text{const}} \{\mathbf{E}[X(1-Z)] - \delta \|(X - \mathbf{E}[X])^-\|_{L^2}\}. && (3.27) \end{aligned}$$

In order to solve the last optimization problem, we construct the Lagrangian function L (where condition "$X \neq$const" can be written as "$\|(X - \mathbf{E}[X])^-\|_{L^2} > 0$") and impose the Kuhn-Tucker optimality conditions $\nabla L = 0$, thus obtaining

$$1 - Z + \delta \cdot \frac{(X - \mathbf{E}[X])^- - \|(X - \mathbf{E}[X])^-\|_{L^1}}{\|(X - \mathbf{E}[X])^-\|_{L^2}} = 0, \tag{3.28}$$

since $\mathbf{E}[Z] = 1$.

Now, for $Z \in \mathcal{Z}$ admitting a payoff $X \in L^\infty$ which solves (3.28), the maximization over non-constant prospects yields zero as result:

$$\begin{aligned} \mathbf{E}[X(1-Z)] - \delta \|(X - \mathbf{E}[X])^-\|_{L^2} &= \mathbf{E}[(X - \mathbf{E}[X])(1-Z)] - \delta \|(X - \mathbf{E}[X])^-\|_{L^2} \\ &= -\frac{\delta \mathbf{E}[(X - \mathbf{E}[X])(X - \mathbf{E}[X])^-]}{\|(X - \mathbf{E}[X])^-\|_{L^2}} - \delta \|(X - \mathbf{E}[X])^-\|_{L^2} \\ &= 0, \end{aligned}$$

so that X solves the problem in (3.26) as well. Therefore, we have that such Z lies in $\text{dom}(V_\delta^2)$ and

$$X \in \arg\max_{\xi \in L^\infty} \{U_\delta^2(\xi) - \mathbf{E}[Z\xi]\}, \tag{3.29}$$

which is equivalent to say $X \in -\partial V_\delta^2(Z)$. On the other hand, if a density Z in $\mathrm{dom}(V_\delta^2)$ does not admit any payoff X which solves (3.28), then $\partial V_\delta^2(Z)$ just contains the constant payoffs. This concludes the proof of (ii) and, by Theorem 2.6, statement (i) holds as well. $\square$

We note that, for any $Z \in \mathcal{Z}$ s.t. there exists $X \in L^\infty$ which solves (3.28), the differential in (3.25) can be rewritten as follows:

$$\partial V_\delta^2(Z) = \Big\{ \frac{\mathbf{E}[Y]}{1-z} Z - Y + c : \, c \in \mathbb{R}, \, Y \in L_+^\infty \text{ and } Y \mathbf{1}_{\{Z \neq z\}} \equiv 0 \Big\}, \tag{3.30}$$

where $L_+^\infty := \{M \in L^\infty : M \geq 0\}$ and $z := \min_\omega Z(\omega) < 1$, with $\mathbb{P}(Z = z) > 0$ and $1 - z = \sqrt{\delta^2 - \mathrm{Var}(Z)}$.

Remark 3.18. *The semi-deviation utility is a classical one-sided measure, hence it results to be a good choice, for example, in calculating the solvency margins. A different situation arises for the standard-deviation principle (3.39) (see Remark 3.25).*

3.3 Non-Monotone Choice Functionals

So far we have considered preference functionals under the classical assumption of monotonicity. Here we drop this assumption in order to include in our study some well known, widely used non-monotone choice criterions: the mean-variance principle and the standard-deviation principle. The fact that such principles are employed in several financial fields (e.g., in the portfolio selection problem as well as in the premium calculation problem), shows how properties often taken for granted as being desirable (as monotonicity for preference criterions) are in fact not so obvious. In particular, the theory we develop for these functionals will allows us to include them in the study of the optimal risk sharing.

3.3.1 A Wider Class of Choice Criterions

The behaviour of an agent endowed with a choice functional failing the property of monotonicity, results, to some extent, in contrast with the economic principle "more is better than less". Therefore, in this case, pathological situations may occur, as clearly shown in Section 3.4.

Remark 3.19. *When monotonicity is lacking, we also lose the property of Lipschitz-continuity with respect to the supremum norm, which is automatically satisfied in the case of monetary utility functionals (see Remark 3.2).*

On the other hand we will see that, in order to develop a theory of risk exchange based on convex analysis, we will need some regularity condition on the involved choice functionals. In view of this, throughout Part I of the thesis we will consider functionals fulfilling the following assumption:

Assumption 3.20. *U is a proper, concave and cash-invariant functional on $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with $U(0) = 0$. Moreover it is continuous with respect to the supremum norm (we equivalently say that it is $L^\infty-$continuous or $\|.\|_{L^\infty}-$continuous).*

All monetary utility functionals automatically satisfy this condition, but, clearly, Assumption 3.20 characterizes a wider family, also containing non-monotone functionals. At this point, for a generic element in this class we cannot directly apply the results stated in Section 3.1 as far as monetary utility functionals are concerned. In spite of that, we obtain similar results which allow us to work in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ instead of the whole dual space $L^\infty(\Omega, \mathcal{F}, \mathbb{P})^*$, as shown in Theorem 3.22.

3.3.2 Some Representation Results

Lemma 3.21. *Let U be a functional satisfying Assumption 3.20, and let $\mathcal{A}_U$ be its acceptance set. Then the following representation holds for its convex conjugate function V (defined as in (3.1)):*

$$V(\mu) = \begin{cases} \sup_{X \in \mathcal{A}_U} \{-\langle \mu, X \rangle\}, & \text{if } \mu(1) = 1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.31}$$

Proof. If $\mu(1) \neq 1$, then $V(\mu) = +\infty$ readily follows from the cash-invariance, as in Section 3.1. Now let μ be such that $\mu(1) = 1$. Clearly $V(\mu)$ is greater than or equal to the supremum in (3.31), as $U \mid_{\mathcal{A}_U} \geq 0$ by definition. On the other hand, $(Y - U(Y)) \in \mathcal{A}_U$ for any Y in L^∞, by cash-invariance, thus giving

$$\sup_{X \in \mathcal{A}_U} \{-\langle \mu, X \rangle\} \geq -\langle \mu, Y - U(Y) \rangle = U(Y) - \langle \mu, Y \rangle, \ \forall Y \in L^\infty,$$

that completes the proof. □

Once again, by $\|.\|_{L^\infty}$-continuity of U, U and V are $\langle L^\infty, (L^\infty)^*\rangle$-conjugate and U has the dual representation (3.2). In addition to this, we can state a result analogous to that given in Theorem 3.3 when dealing with monetary utility functionals. As the intuition suggests, the set $\mathcal{P}$ (equiv. $\mathcal{Z}$) that allows representation (3.3) (equiv. (3.4)) is here replaced by $\mathcal{P}_\sigma$ (equiv. $\mathcal{Z}_\sigma$).

Theorem 3.22. *Let $U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a functional satisfying Assumption 3.20 and $V : L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ its convex conjugate. Then the following statements are equivalent:*

(i) U can be represented by V restricted to the set $\mathcal{P}_\sigma$ (defined in (2.2)):

$$U(X) = \inf_{\mathbb{Q} \in \mathcal{P}_\sigma} \{V(\mathbb{Q}) + \boldsymbol{E}_{\mathbb{Q}}[X]\}, \ \forall X \in L^\infty, \tag{3.32}$$

(ii) U satisfies the Fatou property (see Theorem 3.3).

Once again, by identification of σ-additive measures $\mathbb{Q}$ with their derivatives $\dfrac{d\mathbb{Q}}{d\mathbb{P}}$, we can rewrite representation (3.32) as

$$U(X) = \inf_{Z \in \mathcal{Z}_\sigma} \{V(Z) + \mathbf{E}[ZX]\}, \ \forall\, X \in L^\infty, \tag{3.33}$$

with $\mathcal{Z}_\sigma$ defined in (2.2).

Proof. (ii) $\Rightarrow$ (i): Assume (ii) to hold and define $\widehat{U}(X) := \inf_{Z \in \mathcal{Z}_\sigma} \{V(Z) + \mathbf{E}[ZX]\}$ for any $X \in L^\infty$. Obviously $U \le \widehat{U}$, so that we have only to prove the inverse inequality to obtain the dual representation (3.33). By the translation-invariance property, this is equivalent to show that $U(X - \widehat{U}(X)) \ge 0$, i.e. $(X - \widehat{U}(X)) \in \mathcal{A}_U$, for all $X \in L^\infty$. Suppose, on the contrary, that there exists $\bar{X} \in L^\infty$ such that $(\bar{X} - \widehat{U}(\bar{X})) \notin \mathcal{A}_U$. The Fatou property ensures that the acceptance set $\mathcal{A}_U$ is closed w.r.t. the weak* topology $\sigma(L^\infty, L^1)$, by Proposition 2.10, and we can apply the Hahn-Banach theorem to separate $(\bar{X} - \widehat{U}(\bar{X}))$ from this set. In this way we obtain a continuous linear functional F on $(L^\infty, \sigma(L^\infty, L^1))$ such that

$$\inf_{X \in \mathcal{A}_U} F(X) > F(\bar{X} - \widehat{U}(\bar{X})). \tag{3.34}$$

Since L^1 is the dual of L^∞ when L^∞ is equipped with the $\sigma(L^\infty, L^1)$-topology, the functional F takes on the following form

$$F(X) = \mathbf{E}[XZ], \ \text{ for some } Z \in L^1. \tag{3.35}$$

It follows that $\mathbf{E}[Z] > 0$. Indeed, if $\mathbf{E}[Z] = 0$ then $F(X + c) = F(X)$, for any $X \in L^\infty$, $c \in \mathbb{R}$, thus giving

$$F(\bar{X} - \widehat{U}(\bar{X})) = F(\bar{X} - \widehat{U}(\bar{X}) + (\widehat{U}(\bar{X}) - U(\bar{X}))) \geq \inf_{X \in \mathcal{A}_U} F(X),$$

which is in contradiction with (3.34). On the other hand, if $\mathbf{E}[Z] < 0$ then $F(X + c) = F(X) + c\mathbf{E}[Z]$, for any $X \in L^\infty$, $c \in \mathbb{R}$. In particular, for $X \in \mathcal{A}_U$ and $c \in \mathbb{R}^+$ we have $(X + c) \in \mathcal{A}_U$ and

$$F(X + c) \to -\infty \quad \text{as} \quad c \to +\infty,$$

once again in contradiction with (3.34). This proves that $\mathbf{E}[Z] > 0$, as previously declared, hence making $Z^* := \dfrac{Z}{\mathbf{E}[Z]} \in L^1$ well defined such that $\mathbf{E}[Z^*] = 1$. By Lemma 3.21 we get

$$\begin{aligned} V(Z^*) &= \sup_{X \in \mathcal{A}_U} \{-\mathbf{E}[Z^* X]\} = -\frac{1}{\mathbf{E}[Z]} \inf_{X \in \mathcal{A}_U} \mathbf{E}[ZX] \\ &= -\frac{1}{\mathbf{E}[Z]} \inf_{X \in \mathcal{A}_U} F(X). \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{E}[Z^* \bar{X}] - \widehat{U}(\bar{X}) &= \mathbf{E}[Z^*(\bar{X} - \widehat{U}(\bar{X}))] = \frac{F(\bar{X} - \widehat{U}(\bar{X}))}{\mathbf{E}[Z]} \\ &< \frac{1}{\mathbf{E}[Z]} \inf_{X \in \mathcal{A}_U} F(X) = -V(Z^*), \end{aligned}$$

by (3.34). This gives us $\widehat{U}(\bar{X}) > \mathbf{E}[Z^* \bar{X}] + V(Z^*)$, in contradiction with the definition of $\widehat{U}$, thus showing that $U \leq \widehat{U}$. In this way we get the desired equality $U = \widehat{U}$ on $L^\infty(\mathbb{P})$, which means that (3.33) is satisfied and proves assertion (i).

(i) $\Rightarrow$ (ii): We now assume (3.33) to hold for any essentially bounded random variable. Consider a bounded sequence $(X_n)_n$ on L^∞, converging $\mathbb{P}$-a.s. to some X. The dominated convergence theorem implies that $\mathbf{E}[ZX_n] \to \mathbf{E}[ZX]$ for any $Z \in L^1$ and, in particular, for any $Z \in \mathcal{Z}_\sigma$. Hence, by representation (3.33),

$$\begin{aligned} U(X) &= \inf_{Z \in \mathcal{Z}_\sigma} \{V(Z) + \lim_{n \to \infty} \mathbf{E}[ZX_n]\} \\ &\geq \limsup_{n \to \infty} \inf_{Z \in \mathcal{Z}_\sigma} \{V(Z) + \mathbf{E}[ZX_n]\} \\ &= \limsup_{n \to \infty} U(X_n), \end{aligned}$$

that completes the proof. □

For the most meaningful non-monotone criterions (the mean-variance and standard-deviation principles), we really obtain a stronger result than that of the previous theorem. Indeed, for any payoff $X \in L^\infty$, we will show that the infimum in (3.32) is obtained for some σ-additive measure $\mathbb{Q} = \mathbb{Q}(X)$, hence it is a minimum (see Section 3.4).

Another result for which we can give the adjusted version in this new setting is Theorem 3.5, where the quantile representation is proved under the additional assumption of law-invariance.

Theorem 3.23. *Let U be a law-invariant functional satisfying Assumption 3.20. Then the following representations hold:*

$$U(X) = \inf_{Z \in \mathcal{Z}_\sigma} \Big\{ -\int_0^1 q_{-Z}(t) q_X(t) dt + V(Z) \Big\}, \ \forall X \in L^\infty, \tag{3.36}$$

and

$$V(Z) = \sup_{X \in L^\infty} \Big\{ U(X) + \int_0^1 q_{-Z}(t) q_X(t) dt \Big\} = \sup_{X \in \mathcal{A}_U} \Big\{ \int_0^1 q_{-Z}(t) q_X(t) dt \Big\}, \tag{3.37}$$

for any derivative $Z \in \mathcal{Z}_\sigma$.

We do not present its proof since it is exactly the same as that of Theorem 3.5, where the desired representations readily follow by applying Lemma 3.4 (compare [33]).

3.4 The Mean-Variance and the Standard-Deviation Principles

Born as a method to solve the portfolio selection problem (see the seminal paper of Markovitz [60]), the mean-variance approach is widely used to shape the choices of economic agents when there is uncertainty. An agent with *mean-variance preferences* is characterized by the choice functional $U_\delta^{mv} : L^\infty \to \mathbb{R}$ given by

$$U_\delta^{mv}(X) := \mathbf{E}[X] - \delta \mathrm{Var}(X), \ \text{for all } X \in L^\infty, \tag{3.38}$$

where $\delta > 0$ is an index of the agent's uncertainty aversion. This criterion is a clear expression of the principle of diversification and it doesn't only control the risk on the down side: it also bounds the possible gain on the up side. This leads to anomalous situations where a smaller payoff is preferred to a bigger one, as shown in the following example.

Example 3.24. *Consider two financial positions X and Y as illustrated in the table below:*

states of the world	ω_1	ω_2
probabilities	α	$1-\alpha$
prospect X	0	0
prospect Y	0	y

with $y \in \mathbb{R}^+ = (0, +\infty)$ and $\alpha \in (0,1)$.

In this case, any rational agent should prefer position Y to X, since it gives a higher payoff in any state of nature. However, for an agent endowed with a choice criterion of type (3.38), we have

$$U_\delta^{mv}(X) = 0 \qquad \text{and} \qquad U_\delta^{mv}(Y) = y(1-\alpha)(1-\delta\alpha y).$$

Therefore, for any $y > \dfrac{1}{\delta\alpha}$ we get $U_\delta^{mv}(Y) < 0$, and the mean-variance agent considers X as being strictly better than Y. This means that, if someone offers her a lottery ticket Y with probability $1-\alpha$ of a "too-big" winning y, then the U_δ^{mv}-agent does not accept it. The reason for this arises from the following fact: a big winning clearly increases the mean payoff, but it also makes the financial position more spread out, thereby increasing the variance. The result is: for a sufficiently large value of y, the increment in variability is not compensated by the increment on mean, thus leading the mean-variance agent to refuse the offer.

A similar situation arises when we consider the standard deviation instead of the variance in (3.38), thus obtaining the so called *standard-deviation principle*:

$$U_\delta^{sd}(X) := \mathbf{E}[X] - \delta \mathrm{Var}(X)^{1/2}, \quad \delta > 0. \tag{3.39}$$

The δ intervening here has the same meaning as the δ in the definition of the mean-variance principle, although they have different dimensions. Indeed, the parameter in (3.39) is dimension-free, whereas the dimension of the parameter in (3.38) is 1/currency, in order to have U_δ^{mv} expressed in monetary units. This fact implies that a change of currency also adjusts the parameter appearing in the mean-variance functional, whereas this is not the case for the standard-deviation one. Let us consider two currencies, e.g. \$ and £, with an exchange rate $r > 0$, i.e. $1\$ = r£$. When passing from \$ to £, the parameter in (3.38) has to suffer a change of scale $1/r$: it passes from δ to δ/r. In this way, by indicating in parenthesis the currency in which quantities are expressed, we obtain

$$U_{\delta}^{mv,(\$)}(X^{(\$)}) = rU_{\delta/r}^{mv,(£)}(X^{(£)}),$$

and the amount of required capital does not result to be affected by the choice of currency, which is the desired independence we mentioned in Section 2.3.

Remark 3.25. *At this point, the first comparison that comes to mind is between the semi-deviation utility (3.22) with* $p = 2$*, and the standard-deviation principle (3.39). We consider the same parameter* $\delta \in (0, 1]$ *for both these choice functionals, and point out that they only differ for the discrepancies which contribute to the risk: only the negative ones for* U_{δ}^{2}*, and both negative and positive for* U_{δ}^{sd}*. Once again there is not one definite choice which is the best in any situation. It depends on what we are interested in, and on the type of risk considered. In the problem of economic capital allocation we have to calculate solvency margins and therefore we may prefer a one-sided measure, hence* U_{δ}^{2}*. On the other hand, if we aim at having a choice criterion linked to the stability of the payoffs we might choose a two-sided measure, hence* U_{δ}^{sd}*.*

Let us now go back to the reason why we introduced the standard-deviation principle. Together with the mean-variance criterion, it is the most representative functional in the class of the non-monotone ones. Indeed, as in the preceding case, by means of a simple example we can see that the property of monotonicity is lacking.

Example 3.26. *Consider the same payoffs as in the previous example:*

states of the world	ω_1	ω_2
probabilities	α	$1-\alpha$
prospect X	0	0
prospect Y	0	y

with $y \in \mathbb{R}^+$ *and* $\alpha \in (0, 1)$*.*

In this case we obtain

$$U_\delta^{sd}(X) = 0 \quad \text{and} \quad U_\delta^{sd}(Y) = y\sqrt{1-\alpha}(\sqrt{1-\alpha} - \delta\sqrt{\alpha}).$$

Therefore, for $\alpha \in (0,1)$ such that $\dfrac{1-\alpha}{\alpha} < \delta^2$ we get $U_\delta^{sd}(Y) < 0$, hence the standard-deviation agent strictly prefers Y to X. As for the mean-variance principle, here we find situations in which the agent seems to choose in contradiction with the rational economic behaviour.

On the other hand, in spite of the lack of monotonicity, the functionals U_δ^{mv} and U_δ^{sd} are "good" choice criterions on L^∞ in the sense of Assumption 3.20, as stated in the following proposition.

Proposition 3.27. *The mean-variance principle (3.38) and the standard-deviation principle (3.39) are law-invariant functionals satisfying Assumption 3.20. Moreover, they are both strictly risk-averse conditionally on any event.*

Proof. We readily see that U_δ^{mv} and U_δ^{sd} are both normalized, cash and law invariant. Moreover, dominated convergence readily implies the *Lebesgue property*, which is a regularity condition stronger than the $\|.\|_{L^\infty}$−continuity. Now let us prove concavity, that for the mean-variance criterion is strictly satisfied on $L_0^\infty \cap \text{dom}(U_\delta^{mv})$. To show this, let us fix $\alpha \in (0,1)$ and a pair X, Y of random variables in L^∞ such that $\mathbf{E}[X] = \mathbf{E}[Y] = 0$ and $X \not\equiv Y$. We get

$$(\alpha X + (1-\alpha)Y)^2 \leq \alpha X^2 + (1-\alpha)Y^2,$$

where the strict inequality holds in a non-vanishing set, since X and Y are not identical. Taking the expectation of both sides we obtain:

$$\begin{aligned} \text{Var}(\alpha X + (1-\alpha)Y) &= \mathbf{E}[(\alpha X + (1-\alpha)Y)^2] < \mathbf{E}[\alpha X^2 + (1-\alpha)Y^2] \\ &= \alpha\mathbf{E}[X^2] + (1-\alpha)\mathbf{E}[Y^2] = \alpha\text{Var}(X) + (1-\alpha)\text{Var}(Y), \end{aligned}$$

hence $U_\delta^{mv}(\alpha X + (1-\alpha)Y) > \alpha U_\delta^{mv}(X) + (1-\alpha)U_\delta^{mv}(Y)$, as announced. We now prove concavity for the standard-deviation criterion which, by positive homogeneity property, is equivalent to super-additivity. We want the following inequality to hold:

$$\mathbf{E}[X] - \delta\sqrt{\text{Var}(X)} + \mathbf{E}[Y] - \delta\sqrt{\text{Var}(Y)} \leq \mathbf{E}[X+Y] - \delta\sqrt{\text{Var}(X+Y)},$$

for any $X, Y \in L^\infty$. This means $\sqrt{\mathrm{Var}(X)} + \sqrt{\mathrm{Var}(Y)} \geq \sqrt{\mathrm{Var}(X+Y)}$ or, equivalently,

$$\mathrm{Var}(X) + \mathrm{Var}(Y) + 2\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)} \geq \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y),$$

i.e. $\dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} \leq 1$, which is always true. Then U_δ^{sd} satisfies axioms (sl) and (c), which completes the proof of the first statement. On the other hand, observe that U_δ^{sd} fails strict concavity, since for $X \equiv 0$ and $Y \not\equiv 0$ we obtain $U_\delta^{sd}(\alpha X + (1-\alpha)Y) = \alpha U_\delta^{sd}(X) + (1-\alpha)U_\delta^{sd}(Y)$, by positive homogeneity.

In order to end the proof, there still remains to show that U_δ^{mv} and U_δ^{sd} satisfy property (S). Let us first consider the mean-variance principle. For any $X \in L^\infty$ and $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ and X is not a.s. constant on A, we have

$$\begin{aligned}
U_\delta^{mv}(X\mathbf{1}_{A^c} + \mathbf{E}[X|A]\mathbf{1}_A) &= \mathbf{E}[X] - \delta\mathrm{Var}(X\mathbf{1}_{A^c} + \mathbf{E}[X|A]\mathbf{1}_A) \\
&= \mathbf{E}[X] + \delta\mathbf{E}[X]^2 - \delta\mathbf{E}[(X\mathbf{1}_{A^c} + \mathbf{E}[X|A]\mathbf{1}_A)^2] \\
&= \mathbf{E}[X] + \delta\mathbf{E}[X]^2 - \delta(\mathbf{E}[X^2\mathbf{1}_{A^c}] + \mathbf{E}[\mathbf{E}[X|A]^2\mathbf{1}_A]) \\
&> \mathbf{E}[X] + \delta\mathbf{E}[X]^2 - \delta(\mathbf{E}[X^2\mathbf{1}_{A^c}] + \mathbf{E}[\mathbf{E}[X^2|A]\mathbf{1}_A]) \\
&= \mathbf{E}[X] + \delta\mathbf{E}[X]^2 - \delta(\mathbf{E}[X^2\mathbf{1}_{A^c}] + \mathbf{E}[X^2\mathbf{1}_A]) \\
&= U_\delta^{mv}(X),
\end{aligned}$$

by Jensen's inequality. This shows that U_δ^{mv} satisfies property (S). In particular, we have

$$\mathrm{Var}(X\mathbf{1}_{A^c} + \mathbf{E}[X|A]\mathbf{1}_A) < \mathrm{Var}(X),$$

which, by passing to the square root, implies that

$$U^{sd}(X\mathbf{1}_{A^c} + \mathbf{E}[X|A]\mathbf{1}_A) > U^{sd}(X),$$

thus making the proof complete. □

This places U_δ^{mv} and U_δ^{sd} in the class of functionals which we deal with in the problem of optimal risk sharing. Moreover, the Lebesgue property ensures that the Fatou property holds as well, thus allowing dual representation in L^1, as stated in Theorem 3.22.

We have shown that the standard-deviation functional satisfies super-additivity. On the other hand, we observe that a completely different situation occurs when dealing with the mean-variance criterion. In particular, U_δ^{mv} exhibits sub-linearity on positively correlated risks, when there is no reason to assume that a position compensates the other one. Moreover, it is linear on incorrelated risks and super-linear on negatively correlated risks. In this last situation it seems reasonable that a position may reduce the riskiness of the other one, which brings us back to the concept that a merger does not create the need for extra capital.

3.4.1 The Mean-Variance Principle and the Fenchel Duality

We now study the mean-variance principle and its representability via its dual function $V_\delta^{mv} : L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$, given by:

$$V_\delta^{mv}(\mu) = \sup_{X \in L^\infty} \{U_\delta^{mv}(X) - \langle \mu, X \rangle\}, \quad \text{for any } \mu \in (L^\infty)^*. \tag{3.40}$$

According to the Fenchel-Moreau theorem, U_δ^{mv} and V_δ^{mv} are $\langle L^\infty, (L^\infty)^* \rangle-$conjugate and, as before pointed out, the Fatou property implies representability in L^1:

$$U_\delta^{mv}(X) = \inf_{Z \in \mathcal{Z}_\sigma} \{V_\delta^{mv}(Z) + \mathbf{E}[ZX]\}, \quad \text{for any } X \in L^\infty. \tag{3.41}$$

We will prove a stronger result: the infimum in (3.41) is attained in L^1 and therefore it is a minimum. This means that, for every $X \in L^\infty$, there exists a signed measure $\mathbb{Q} \in \mathcal{P}_\sigma$ which verifies the equality $U_\delta^{mv}(X) = V_\delta^{mv}(Z) + \mathbf{E}[ZX]$, where Z is the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$. Using tools of calculus of variations, we prove this fact and provide an explicit formulation for the Fenchel transform V_δ^{mv}.

Theorem 3.28. *Let $U_\delta^{mv} : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be the mean-variance functional defined in (3.38), and V_δ^{mv} its Fenchel-Legendre transform (3.40). Then*

(i) $V_\delta^{mv}(Z) = \max_{X \in L^\infty} \{U_\delta^{mv}(X) - \boldsymbol{E}[ZX]\} = \frac{VarZ}{4\delta}, \ \forall Z \in dom\ (V_\delta^{mv}) = \mathcal{Z}_\sigma \cap L^2,$

(ii) $\partial U_\delta^{mv}(X) = \{1 - 2\delta(X - \boldsymbol{E}[X])\}, \ \forall X \in L^\infty,$

(iii) $\partial V_\delta^{mv}(Z) = \left\{\frac{Z}{2\delta} + c, \forall c \in \mathbb{R}\right\}, \ \forall Z \in dom\ (V_\delta^{mv}).$

In particular the following representation holds for U_δ^{mv}:

$$U_\delta^{mv}(X) = \min_{Z \in \mathcal{Z}_\sigma} \{V_\delta^{mv}(Z) + \mathbf{E}[ZX]\} = \min_{Z \in \mathcal{Z}_\sigma} \left\{\frac{\mathbf{E}[Z^2]}{4\delta} + \mathbf{E}[ZX]\right\} - \frac{1}{4\delta}, \ \forall X \in L^\infty. \tag{3.42}$$

Proof. Since U_δ^{mv} is concave and continuous w.r.t. the supremum norm, $\partial U_\delta^{mv}(X)$ is not empty for each X in L^∞, by Theorem 2.5. Let us now prove that it is in fact a singleton for any $X \in L^\infty$, by Gateaux differentiability and Theorem 2.8. For any pair of prospects $X, Y \in L^\infty$ and $\epsilon > 0$, we have

$$\begin{aligned} U_\delta^{mv}(X+\epsilon Y) &= \mathbf{E}[X+\epsilon Y] - \delta \mathrm{Var}(X+\epsilon Y) \\ &= \mathbf{E}[X] + \epsilon \mathbf{E}[Y] - \delta(\mathrm{Var}(X) + \epsilon^2 \mathrm{Var}(Y) + 2\mathrm{Cov}(X, \epsilon Y)) \\ &= U_\delta^{mv}(X) + \epsilon \mathbf{E}[Y] - \delta(\epsilon^2 \mathrm{Var}(Y) + 2\epsilon \mathrm{Cov}(X,Y)) \end{aligned}$$

and therefore, by (2.11), the Gateaux differential of U_δ^{mv} at X with perturbation Y is given by

$$\begin{aligned} J_\delta(X;Y) &:= \lim_{\epsilon \to 0} \frac{U_\delta^{mv}(X+\epsilon Y) - U_\delta^{mv}(X)}{\epsilon} = \mathbf{E}[Y] - 2\delta \mathrm{Cov}(X,Y)) \\ &= \mathbf{E}[Y(1 - 2\delta(X - \mathbf{E}[X]))]. \end{aligned}$$

As seen in Section 2.2, linearity of the functional $J_\delta(X;.) = \nabla U_\delta^{mv}(X)$ means the differentiability of U_δ^{mv} with

$$\partial U_\delta^{mv}(X) = \{\nabla U_\delta^{mv}(X)\} = \{1 - 2\delta(X - \mathbf{E}[X])\}, \tag{3.43}$$

which is exactly (ii). Here we have identified the linear functional $\nabla U_\delta^{mv}(X)$ on L^∞, with the Radon-Nikodym derivative

$$Z_X := 1 - 2\delta(X - \mathbf{E}[X]) \in L^1, \quad \mathbf{E}[Z_X] = 1.$$

At this point, strict concavity of U_δ^{mv} on L_0^∞ implies the existence of a unique gradient of V_δ^{mv} on L_0^∞ for any element in the domain of V_δ^{mv}, by Theorem 2.9. That is, for any $\mu \in \mathrm{dom}(V_\delta^{mv})$, there exists $X_\mu \in L^\infty$ such that $\partial V_\delta^{mv}(\mu) = \{X_\mu + c : c \in \mathbb{R}\}$. By Theorem 2.6 we know that $X_\mu \in \partial V_\delta^{mv}(\mu)$ implies $\mu \in \partial U_\delta^{mv}(-X_\mu)$ and, by (3.43), we get $\mu \in \mathcal{Z}_\sigma$. This fact leads to

$$\mathrm{dom}(V_\delta^{mv}) \subseteq \mathcal{Z}_\sigma$$

and

$$\partial V_\delta^{mv}(Z) = \left\{\frac{Z}{2\delta} + c : c \in \mathbb{R}\right\}, \ \forall Z \in \mathrm{dom}(V_\delta^{mv}),$$

as stated in (iii). Moreover we have

$$\partial V_\delta^{mv}(Z) = -\arg\max_{X\in L^\infty} \{U_\delta^{mv}(X) - \mathbf{E}[ZX]\},$$

by (2.10), and for any $X_Z \in -\partial V_\delta^{mv}(Z)$ we obtain

$$\begin{aligned} V_\delta^{mv}(Z) &= U_\delta^{mv}(X_Z) - \mathbf{E}[ZX_Z] \;=\; \mathbf{E}[X_Z(1-Z)] - \delta \mathrm{Var}(X_Z) \\ &= \mathbf{E}\Big[-\frac{Z}{2\delta}(1-Z)\Big] - \delta Var\Big(-\frac{Z}{2\delta}\Big) \\ &= \frac{\mathrm{Var}(Z)}{4\delta}, \end{aligned}$$

which shows (i) and completes the proof. □

3.4.2 The Standard-Deviation Principle and the Fenchel Duality

In the same way we study the standard-deviation principle, obtaining the following result:

Theorem 3.29. *Let $U_\delta^{sd} : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be the standard-deviation principle defined in (3.39) and $V_\delta^{sd} : L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0,\infty]$ its convex conjugate function. Then*

(i) for any $X \in L^\infty$,

$$\partial U_\delta^{sd}(X) = \begin{cases} dom(V_\delta^{sd}), & \text{if } X = const, \\ \Big\{1 - \delta\dfrac{X - \boldsymbol{E}[X]}{\sqrt{Var(X)}}\Big\}, & \text{otherwise}, \end{cases}$$

(ii) for any $Z \in dom\ (V_\delta^{sd})$,

$$\partial V_\delta^{sd}(Z) = \{c :\ c \in \mathbb{R}\} \cup \Big\{X \in L^\infty : Z = 1 + \delta\frac{X - \boldsymbol{E}[X]}{\sqrt{Var(X)}}\Big\}.$$

In particular, $\Big\{1 - \delta\dfrac{X - \mathbf{E}[X]}{\sqrt{\mathrm{Var}(X)}} \;:\; X \in L^\infty \setminus \{c :\ c \in \mathbb{R}\}\Big\}$ is the minimal set C which allows us to represent U_δ^{sd} in the following form:

$$U_\delta^{sd}(X) = \min_{Z\in C} \mathbf{E}[ZX],\ \forall X \in L^\infty.$$

Proof. Since U_δ^{sd} is positively homogeneous, we know that V_δ^{sd} is equal to zero on its domain and, in particular, for any $Z \in \mathcal{Z}_\sigma$ we have

$$\begin{aligned} V_\delta^{sd}(Z) &= \sup_{X \in L^\infty} \{\mathbf{E}[X(1-Z)] - \delta \|X - \mathbf{E}[X]\|_{L^2}\} && (3.44) \\ &= 0 \vee \sup_{X \in L^\infty, X \neq \text{const}} \{\mathbf{E}[X(1-Z)] - \delta \|X - \mathbf{E}[X]\|_{L^2}\}. && (3.45) \end{aligned}$$

As in the case of the semi-deviation utility, we solve the last optimization problem by constructing the Lagrangian function L (where "$X \neq$ const" can be written as "$\|X - \mathbf{E}[X]\|_{L^2} >$ 0") and imposing the Kuhn-Tucker optimality conditions $\nabla L = 0$. In this way we obtain

$$1 - Z - \delta \cdot \frac{X - \mathbf{E}[X]}{\sqrt{\mathrm{Var}(X)}} = 0. \tag{3.46}$$

Therefore, if $Z \in \mathcal{Z}_\sigma$ admits a payoff X which solves (3.46), then $Z \in L^\infty$ with $\mathrm{Var}(Z) = \delta^2$, and the maximization over non-constant prospects yields zero as result:

$$\begin{aligned} \mathbf{E}[X(1-Z)] - \delta\sqrt{\mathrm{Var}(X)} &= \mathbf{E}[(X - \mathbf{E}[X])(1-Z)] - \delta\sqrt{\mathrm{Var}(X)} \\ &= \frac{\delta}{\sqrt{\mathrm{Var}(X)}} \mathbf{E}[(X - \mathbf{E}[X])^2] - \delta\sqrt{\mathrm{Var}(X)} = 0. \end{aligned}$$

From this fact it follows that X solves the problem in (3.44) as well, that is

$$X \in \arg\max_{\xi \in L^\infty} \{U_\delta^{sd}(\xi) - \mathbf{E}[Z\xi]\}, \tag{3.47}$$

which means that $X \in -\partial V_\delta^{sd}(Z)$. Note that in this case we can write any solution of (3.44) as $X = d_1 Z + d_2$, with $d_1 \leq 0$ and $d_2 \in \mathbb{R}$. On the other hand, if an element Z in $\mathrm{dom}(V_\delta^{sd})$ does not admit any payoff X which solves (3.46), then $\partial V_\delta^{sd}(Z)$ just contains the constant payoffs. This reasoning shows (*ii*) and, by Theorem 2.6, statement (i) is also proved. □

3.5 From Non-Monotone to Monotone Criterions

For any element in the class of criterions characterized by Assumption 3.20, we want to give the "best possible" approximation in the smaller class of monetary utility functionals, where the axiom of monotonicity is satisfied. This is the unique property we have to get back and therefore our "correction" goes in this direction. In order to do this, we follow the same reasoning as Maccheroni, Marinacci, Rustichini and Taboga [59], where the *best* *monotone approximation* of non-monotone preferences is introduced to solve a portfolio selection problem.

First we characterize the set of financial positions in L^∞ where a non-monotone agent behaves monotonically.

Definition 3.30 ([59]). *Let $U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a functional satisfying Assumption 3.20. We call domain of monotonicity of U the following subset of L^∞:*

$$M(U) := \{X \in L^\infty : \partial U(X) \cap (L^\infty)^*_+ \neq \emptyset\}, \tag{3.48}$$

where $(L^\infty)^*_+$ is the collection of all positive measures of $(L^\infty)^*$.

$M(U)$ consists of all random variables in L^∞ where U admits al least one positive gradient and, as this definition suggests, functional U restricted to $M(U)$ is monotone. Indeed, let $M(U) \neq \emptyset$ and $X, Y \in M(U)$ such that $X \leq Y$. We prove that $U(X) \leq U(Y)$. By hypothesis there exists $\mu_Y \in \partial U(Y) \cap (L^\infty)^*_+$, which gives us $\langle \mu_Y, \xi \rangle \geq 0$ whenever $\xi \geq 0$. By the Fenchel-Moreau theorem we get

$$\begin{aligned} U(X) &= \inf_{\mu \in (L^\infty)^*} \{V(\mu) + \langle \mu, X \rangle\} \leq V(\mu_Y) + \langle \mu_Y, X \rangle \\ &= V(\mu_Y) + \langle \mu_Y, Y \rangle - \langle \mu_Y, Y - X \rangle = U(Y) - \langle \mu_Y, Y - X \rangle \\ &\leq U(Y), \end{aligned}$$

as desired.

Consider, for example, the mean-variance principle U_δ^{mv}. From (3.43) we obtain

$$M(U_\delta^{mv}) = \{X \in L^\infty : \nabla U_\delta^{mv}(X) \in \mathcal{Z}\} = \Big\{X \in L^\infty : X - \mathbf{E}[X] \leq \frac{1}{2\delta}\Big\}.$$

This means that the mean-variance functional is monotone on the set of prospects with sufficiently small variability in the right hand side of the distribution, that is, when the highest values of the payoff are not too spread out. On the other hand, for the standard-deviation principle U_δ^{sd}, we have

$$M(U_\delta^{sd}) = \{X \in L^\infty : \partial U_\delta^{sd} \cap \mathcal{Z} \neq \emptyset\} = \{X = \text{const}\} \cup \Big\{X \in L^\infty : \frac{X - \mathbf{E}[X]}{\sqrt{\text{Var}(X)}} \leq \frac{1}{\delta}\Big\}.$$

This indicates that U_δ^{sd} is monotone on the set of prospects with standardized version bounded from above. Therefore, we don't want the variability of the right hand side of the distribution to be too high with respect to the variance of the entire payoff.

At this point, we introduce the same monotone adjusted-version proposed in [59]: for any non-monotone criterion we consider the monotone functional that coincides with the former on its domain of monotonicity and that better approximates it among all monetary utility functionals. In this purpose, let us fix a functional $U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ satisfying Assumption 3.20, with $M(U) \neq \emptyset$. We can observe that, for example, this is the case when U is law-invariant, since we have $V(1) = 0$ (compare [49]), $1 \in \partial U(c)$ for any $c \in \mathbb{R}$, hence $\{c : c \in \mathbb{R}\} \subseteq M(U)$. The idea is to define a m.u.f. U^m by means of the convex conjugate $V : L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ of U. Calling to mind that U can be represented in the following way:

$$U(X) = \inf_{\mu \in (L^\infty)^*} \{V(\mu) + \langle \mu, X \rangle\}, \ \forall X \in L^\infty,$$

we consider the monetary utility functional $U^m : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ given by

$$U^m(X) := \inf_{\mu \in (L^\infty)^*_+} \{V(\mu) + \langle \mu, X \rangle\}, \ \forall X \in L^\infty. \tag{3.49}$$

Theorem 3.31 ([59]). *Let $U^m : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be the monetary utility functional defined in (3.49). Then the following assertions hold:*

(i) U^m is the minimal monotone functional that dominates U, that is

$$U^m(X) = \sup\{U(Y) : Y \in L^\infty \text{ and } Y \leq X\}, \ \forall X \in L^\infty, \tag{3.50}$$

(ii) for any $X \in L^\infty$, $X \in M(U)$ if and only if $U(X) = U^m(X)$,

(iii) let $V^m : L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ be the dual conjugate of U^m and $\mu \in (L^\infty)^*$, then

$$V^m(\mu) = \begin{cases} V(\mu), & \text{if } \mu \in (L^\infty)^*_+, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.51}$$

This theorem states that U^m is the most conservative monetary utility functional that extends U outside its domain of monotonicity, and in this sense it is the best possible monotone approximation.

Representation (3.50) of U^m is in full agreement with the intuition and with the usual procedures in utility maximization. We look for the highest possible level of satisfaction we can obtain from a given payoff, and therefore if we can benefit from withdrawing money from our portfolio, then we do it. By Proposition 3.32 we see that in the case of mean-variance preferences this corresponds to a truncation from above of the payoffs.

Recalling that the mean-variance principle U_δ^{mv} can be written as

$$U_\delta^{mv}(X) = \min_{Z\in\mathcal{Z}_\sigma}\{V_\delta^{mv}(Z) + \mathbf{E}[ZX]\} = \min_{Z\in\mathcal{Z}_\sigma}\Big\{\frac{\mathrm{Var}(Z)}{4\delta} + \mathbf{E}[ZX]\Big\},$$

its natural corrected version (the *monotone mean-variance principle*) is given by

$$U_\delta^{mmv}(X) := \inf_{Z\in\mathcal{Z}}\Big\{\frac{\mathrm{Var}(Z)}{4\delta} + \mathbf{E}[ZX]\Big\},\ \forall X\in L^\infty. \tag{3.52}$$

Moreover, the infimum in (3.52) is attained in $\mathcal{Z}$, since U_δ^{mmv} satisfies the Lebesgue property, hence it is a minimum.

Proposition 3.32 ([59]). *Let U_δ^{mv} be the mean-variance criterion and U_δ^{mmv} its best monotone approximation. Then*

$$U_\delta^{mmv}(X) = \begin{cases} U_\delta^{mv}(X), & \text{if } X\in M(U_\delta^{mv}),\\ U_\delta^{mv}(X\wedge k_X), & \text{otherwise,}\end{cases}$$

where $k_X = \max\{t\in\mathbb{R} : X\wedge t\in M(U_\delta^{mv})\}$.

Example 3.33. *Consider the same payoffs as in Example 3.24. We have $Y\in M(U_\delta^{mv})$ if and only if*

$$0-(1-\alpha)y \le \frac{1}{2\delta} \qquad \text{and} \qquad y-(1-\alpha)y\le\frac{1}{2\delta},$$

that is, if and only if

$$y\le\frac{1}{2\delta\alpha}.$$

Therefore, for a prospect Y of this shape, we have truncation at level $k_Y = \dfrac{1}{2\delta\alpha}$.

Once again, let us consider the interpretation of Y as a lottery ticket. Whereas we have seen that a U_δ^{mv}-agent only accepts poor-lottery tickets, a U_δ^{mmv}-agent behaves in a more rational way. Indeed, she accepts tickets of any lottery, even if she considers lottery tickets with winnings $y\ge k_Y$ as being equivalent.

A different situation arises when considering the lottery example for the standard-deviation principle.

Example 3.34. *Consider the same payoffs as in Example 3.26. Here we get* $Y \in M(U_\delta^{sd})$ *if and only if*

$$\frac{0-(1-\alpha)y}{y\sqrt{\alpha(1-\alpha)}} \leq \frac{1}{\delta} \qquad \textit{and} \qquad \frac{y-(1-\alpha)y}{y\sqrt{\alpha(1-\alpha)}} \leq \frac{1}{\delta},$$

that is, if and only if

$$\frac{\alpha}{1-\alpha} \leq \frac{1}{\delta^2},$$

independently of the winning y.

Chapter 4

Optimal Sharing of Aggregate Risks

Consider an aggregate of n economic agents, for some $n \in \mathbb{N}$, endowed with initial risks $\xi_1, \dots, \xi_n \in L^\infty$ and characterized by choice functionals $U_1, \dots, U_n$ defined on L^∞. This means that, if today their situation remains unchanged, then tomorrow agent i will have to face her risky position ξ_i. Now the question that arises is if the agents may re-share the total risk

$$X = \sum_{i=1}^{n} \xi_i$$

in order to make their situation better. Clearly here "better" has the meaning of "more satisfactory" in the sense of the choice criterions $(U_i)_{i=1}^n$, hence this problem is directly linked to the agents' preferences.

As we have said in the Introduction, this concept captures various situations with very different characteristics and purposes. Among these it's worthy of note the case of risk exchange in insurance and reinsurance contracts, a problem investigated since the early work of Arrow [2] and Borch [9] (see also [38], [39], [12]). It's fundamental to note that we can indistinctly consider any case in which there is an aggregate of economic agents willing to enter into a contract to better their position. The distinction will be made when considering the respective initial risk endowments and, above all, when choosing the appropriate preference criterions for the involved agents.

In this chapter we study the optimal sharing of aggregate risks with a double purpose: collective and individual satisfaction. First, we aim at maximizing the sum of the utilities that agents assign at their own share of risk, i.e., we consider the optimization problem (4.1) which jointly involves all agents. Successively, we take into consideration the individual point of view of each agent, looking for a contract that everyone agrees to sign. Basically, we extend the results of Jouini et al. [49] on the existence and the characterization of optimal solutions to these problems, to a more general setting. In the above-mentioned paper, two agents with monotone preferences are considered. Here we deal with any number of agents endowed with more general choice criterions, as those introduced in the previous chapter. This will also allow us to compare the behaviour of monotone and non-monotone agents in facing the risk sharing problem and to show that, under suitable conditions, the optimal redistribution of a given aggregate risk is not affected by the presence of agents failing monotonicity property.

4.1 The Sup-Convolution Problem

To formalize our problem, we introduce the set of admissible sharings of an essentially bounded risk.

Definition 4.1. *Given an aggregate of n agents and a risk $X \in L^\infty$, we define the set of attainable allocations as the following collection of n-tuples:*

$$\mathbb{A}_n(X) := \{(X_1, \dots, X_n) \in L^\infty \times \cdots \times L^\infty : \sum_{i=1}^{n} X_i = X\}.$$

Since it seems to be reasonable to require each quota of risk to rise with the total risk, we also introduce the set of admissible allocations which increase with the corresponding aggregate risk:

$$\mathbb{A}_n^{\uparrow}(X) := \{(X_1, \dots, X_n) \in \mathbb{A}_n(X) : X_i \nearrow X, \ \forall i = 1, \dots, n\},$$

where $X_i \nearrow X$ means $X_i = \phi_i(X)$ pointwise, for some non-decreasing function $\phi_i : \mathbb{R} \to \mathbb{R}$.

As pointed out before, we first consider the "common welfare" by solving the sup-convolution problem:

$$U_1 \square \cdots \square U_n(X) := \sup_{(X_1,..,X_n)\in \mathbb{A}_n(X)} \sum_{i=1}^n U_i(X_i), \tag{4.1}$$

which provides solutions jointly optimal for the agents, without regard to their individual level of satisfaction.

Remark 4.2. *We essentially deal with choice functionals U_i characterized in Assumption 3.20, which, in general, don't satisfy monotonicity. However we often refer to $U_i(X_i)$ as the utility that agent i obtains from position X_i, and to $\sum_{i=1}^n U_i(X_i)$ as the overall utility.*

We will see that the solutions to the maximization problem (4.1), if they exist, provide such allocations which do not allow a better position for all agents. Nevertheless, the re-sharing suggested by a solution of (4.1) may worsen some agents' situation, thus making the introduction of some constraints on such an optimization problem necessary, in order to avoid these unpleasant occurrences. Before that, we must fix some notations and simple results linked to the sup-convolution problem.

Here and throughout this chapter, we consider a collection of choice functionals $U_1, \dots, U_n$ satisfying Assumption 3.20. By means of these, we define another functional U on L^∞, which is the result of their sup-convolution:

$$U(X) := U_1 \square \cdots \square U_n(X), \quad \text{for any } X \in L^\infty. \tag{4.2}$$

Since $U_i : L^\infty \to \mathbb{R}$ are concave and finite functionals, then U is also concave and satisfies $U > -\infty$ on L^∞. In this way we get $U : L^\infty \to \mathbb{R} \cup \{+\infty\}$, with either $U \equiv +\infty$ or $\text{dom}(U) = L^\infty$, from the concavity property.

Denote by $V, V_1, \dots, V_n : (L^\infty)^* \to [0, +\infty]$ the convex conjugate functions of $U, U_1, \dots, U_n$ respectively. Then we obtain $V \equiv +\infty$ if $U \equiv +\infty$, and

$$V = \sum_{i=1}^n V_i, \quad \text{with} \quad \text{dom}(V) = \bigcap_{i=1}^n \text{dom}(V_i), \tag{4.3}$$

if U is proper. Indeed, for any $\mu \in (L^\infty)^*$, we have

$$\begin{aligned} V(\mu) &= \sup_{X\in L^\infty} \{U(X) - \langle \mu, X\rangle\} = \sup_{X\in L^\infty} \sup_{(X_i)_i \in \mathbb{A}_n(X)} \left\{ \sum_{i=1}^n U_i(X_i) - \langle \mu, X\rangle \right\} \\ &= \sup_{X_i \in L^\infty} \sum_{i=1}^n (U_i(X_i) - \langle \mu, X_i\rangle) = \sum_{i=1}^n V_i(\mu). \end{aligned}$$

To avoid the worthless case $U \equiv +\infty$, we make the following assumption on the dual functions V_i's, which is equivalent to have U proper with $\text{dom}(U) = L^\infty$.

Assumption 4.3. *The convex conjugate functions $V_1, \dots, V_n$ are s.t.* $\bigcap_{i=1}^{n} dom(V_i) \neq \emptyset$.

Under this condition we study some of the properties U inherits from functionals U_i's. Cash-invariance is immediate. Moreover, since $\text{dom}(U) = L^\infty$, Theorem 2.7 ensures that $\partial U(X) \neq \emptyset$ for all $X \in L^\infty$, and that (U, V) are $\langle L^\infty, (L^\infty)^* \rangle-$ conjugate.

Remark 4.4. *We stress the fact that, under law-invariance property for U_i's, Assumption 4.3 is automatically satisfied, seeing that $Z \equiv 1$ lies in the effective domain of V_i, with $V_i(1) = 0$, for any $i = 1, \dots, n$ (compare [49]). This also guarantees the normalization property for U, by relation (4.3).*

In the following lemma we present further stability properties for our preference criterions.

Lemma 4.5. *Let $(U_i)_{i=1}^n$ be choice criterions satisfying Assumption 3.20 and Assumption 4.3, and let U be the functional defined in (4.2). Then the following implications hold:*

(i) U_j *monotone for some* $j \in \{1, \dots, n\} \;\Rightarrow\; U$ *monotone,*

(ii) $(U_i)_{i=1}^n$ *law-invariant* $\;\Rightarrow\; U$ *law-invariant,*

(iii) $(U_i)_{i=1}^n$ *law-invariant and satisfy property* $(S) \;\Rightarrow\; U$ *satisfies property* (S)*,*

(iv) $(U_i)_{i=1}^n$ *law-invariant and* U_j *strictly monotone for some* $j \in \{1, \dots, n\} \;\Rightarrow\; U$ *strictly monotone.*

On the contrary, as seen in a counter-example reported in [22], we have that the Fatou property is not stable for the sup-convolution.

Proof. [Lemma 4.5-(i),(ii)] Statement (ii) being immediate, we assume U_j to be monotone for some $j \in \{1, \dots, n\}$ and prove statement (i). To show that U is monotone as well, let us consider two prospects $X, Y \in L^\infty$ such that $X \leq Y$. Let $(X_1^m, \dots, X_n^m)_{m \in \mathbb{N}}$ be a maximizing sequence in $\mathbb{A}_n(X)$ for the sup-convoultion problem, that is $U(X) = \lim_{m \to +\infty} \sum_{i=1}^{n} U_i(X_i^m)$, and define the allocations $(Y_1^m, \dots, Y_n^m)_{m \in \mathbb{N}}$ in $\mathbb{A}_n(Y)$ given by

$$Y_i^m = \begin{cases} X_j^m + (Y - X), & \text{if } i = j, \\ X_i^m, & \text{if } i \neq j. \end{cases}$$

At this point, by definition of U and monotonicity of U_j we have

$$U(Y) \geq \lim_{m \to +\infty} \sum_{i=1}^{n} U_i(Y_i^m) \geq \lim_{m \to +\infty} \sum_{i=1}^{n} U_i(X_i^m) = U(X).$$

This makes U a monetary utility functional and, in particular, L^∞-Lipschitz continuous. □

Proof of statements (iii)-(iv) will be given after Theorem 4.11.

4.1.1 Characterization of Pareto Optimal Allocations

We now study the sup-convolution problem, showing that it is the recipe to find the Pareto optimal allocations, in the sense of the definition below.

Definition 4.6. *An n-tuple $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is said a Pareto Optimal Allocation (POA) if for all $(\xi_1, \dots, \xi_n) \in \mathbb{A}_n(X)$ such that $U_i(\xi_i) \geq U_i(X_i)\ \forall\, i = 1, \dots, n$, then $U_i(\xi_i) = U_i(X_i)\ \forall\, i = 1, \dots, n$.*

This means that there exists no other redistribution of the aggregate risk X such that each agent is better off with at least one strictly better.

Remark 4.7. *Due to the translation invariance, Pareto optimality is true up to constants summing up to zero. That is, for any POA $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ of a given aggregate risk X, and any choice of constants $c_1, \dots, c_n \in \mathbb{R}$ s.t. $\sum_{i=1}^{n} c_i = 0$, then also $(X_1 + c_1, \dots, X_n + c_n)$ is a POA of X.*

The following theorem provides a characterization of Pareto optimal allocations as solutions to the sup-convolution problem and by means of convex analysis tools (see [49, Theorem 3.1] for the case of two agents endowed with monetary utility functionals).

Theorem 4.8. *Let $U_1, \dots, U_n$ be preference functionals satisfying Assumption 3.20 and Assumption 4.3, with associated dual convex functions $V_1, \dots, V_n$ defined on $(L^\infty)^*$. For a given risk $X \in L^\infty$ and an allocation $(\xi_1, \dots, \xi_n) \in \mathbb{A}_n(X)$, the following statements are equivalent:*

(i) $(\xi_1, \dots, \xi_n)$ *is a Pareto optimal allocation,*

(ii) $U_1 \square \cdots \square U_n(X) = \sum_{i=1}^{n} U_i(\xi_i)$,

(iii) there exists $\mu \in (L^\infty)^*$ *s.t.* $U_i(\xi_i) = V_i(\mu) + \langle \mu, \xi_i \rangle \;\; \forall i = 1, \dots, n$,

(iv) there exists $\mu \in (L^\infty)^*$ *s.t.* $\mu \in \partial U_i(\xi_i) \;\; \forall i = 1, \dots, n$,

(v) there exists $\mu \in (L^\infty)^*$ *s.t.* $-\xi_i \in \partial V_i(\mu) \;\; \forall i = 1, \dots, n$.

Proof. (ii) $\Rightarrow$ (i) is trivial, and (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) directly follow from Theorem 2.6, since (U_i, V_i) are $\langle L^\infty, (L^\infty)^* \rangle-$conjugate for each i. Let us prove the other implications.
(i) $\Rightarrow$ (ii): Let's assume (i) and $\rceil$(ii), that is, $(\xi_1, \dots, \xi_n)$ POA with $\sum_{i=1}^{n} U_i(\xi_i) < \sum_{i=1}^{n} U_i(\zeta_i)$, for some $(\zeta_1, \dots, \zeta_n) \in \mathbb{A}_n(X)$. Define

$$\alpha_i := U_i(\zeta_i) - U_i(\xi_i), \; \forall i = 1, \dots, n \quad \text{and} \quad \alpha := \sum_{i=1}^{n} \alpha_i > 0.$$

Put $I := \{1, \dots, n\}, \; J := \{j \in I : \alpha_j \neq 0\} \neq \emptyset$ and let $m > 0$ be the cardinality of J. The cash can be rebalanced to obtain the allocation $(X_1, \dots, X_n)$ as follows:

$$X_i = \begin{cases} \zeta_i, & \text{for all } i \in I \setminus J, \\ \zeta_i - \alpha_i + \dfrac{\alpha}{m}, & \text{for all } i \in J. \end{cases}$$

Obviously $\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \zeta_i = X$, so that $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$. Moreover we get

$$U_i(X_i) = \begin{cases} U_i(\xi_i), & \text{for all } i \in I \setminus J, \\ U_i(\xi_i) + \dfrac{\alpha}{m} > U_i(\xi_i), & \text{for all } i \in J, \end{cases}$$

which is in contradiction with the Pareto optimality of $(\xi_1, \dots, \xi_n)$, and therefore (ii) holds whenever (i) holds.
(ii) $\Rightarrow$ (iii): Recall that, under our assumptions, $\partial U(X) \neq \emptyset \;\; \forall X \in L^\infty$. At this point there exists a measure $\mu \in (L^\infty)^*$ such that $U(X) = V(\mu) + \langle \mu, X \rangle$, and from (ii) and (4.3) we obtain

$$\sum_{i=1}^{n} U_i(\xi_i) = \sum_{i=1}^{n} V_i(\mu) + \langle \mu, \sum_{i=1}^{n} \xi_i \rangle. \tag{4.4}$$

On the other hand, by duality relations we have

$$U_i(\xi_i) \le V_i(\mu) + \langle \mu, \xi_i \rangle, \text{ for all } i = 1, \dots, n,$$

hence (iii) follows from (4.4).
(iii) ⇒ (ii): Clearly $\sum_{i=1}^n U_i(\xi_i) = \sum_{i=1}^n V_i(\mu) + \langle \mu, \sum_{i=1}^n \xi_i \rangle = V(\mu) + \langle \mu, X \rangle$. By duality relation and (4.2), it follows that $U(X) \le V(\mu) + \langle \mu, X \rangle = \sum_{i=1}^n U_i(\xi_i) \le U(X)$, which implies (ii) and completes the proof. □

Proposition 4.9. *Let U be defined in (4.2). For any $X \in L^\infty$, the following implications hold true:*

(i) $\mu \in \partial U(X) \;\Rightarrow\; \mu \in \bigcap_{i=1}^n \partial U_i(X_i)$ *for any POA* $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$,

(ii) $\mu \in \bigcap_{i=1}^n \partial U_i(X_i)$ *for some* $(X_i)_{i=1}^n \in \mathbb{A}_n(X) \;\Rightarrow\; \mu \in \partial U(X)$.

Proof. (i): Let $\mu \in \partial U(X)$ and let $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$ be a POA of X. By the Fenchel duality relation we have

$$U(X) = V(\mu) + \langle \mu, X \rangle = \sum_{i=1}^n V_i(\mu) + \langle \mu, \sum_{i=1}^n X_i \rangle \ge \sum_{i=1}^n U_i(X_i), \tag{4.5}$$

as $V_i(\mu) + \langle \mu, X_i \rangle \ge U_i(X_i)$ for any $i = 1, \dots, n$. Now, since $U(X) = \sum_{i=1}^n U_i(X_i)$, the inequality in (4.5) results to be an equality and therefore $V_i(\mu) + \langle \mu, X_i \rangle = U_i(X_i)$ for any i=1,. . . ,n, which is the stated result.
(ii): Let $(X_i)_{i=1}^n$ be an admissible allocation of X and μ a measure such that $\mu \in \partial U_i(X_i)$ $\forall i = 1, \dots, n$. Theorem 4.8 implies that $(X_i)_{i=1}^n$ is a POA and we obtain

$$U(X) = \sum_{i=1}^n U_i(X_i) = \sum_{i=1}^n V_i(\mu) + \langle \mu, \sum_{i=1}^n X_i \rangle = V(\mu) + \langle \mu, X \rangle,$$

which ensures that μ lies in $\partial U(X)$ and makes the proof complete.

□

The following lemma shows that the problem of sup-convolution can be given in terms of allocations increasing with the total risk.

Lemma 4.10. *Let $U_1, \ldots, U_n$ be law-invariant choice criterions satisfying Assumption 3.20. Then*

$$\sup_{(X_i)_{i=1}^n \in \mathbb{A}_n(X)} \sum_{i=1}^n U_i(X_i) = \sup_{(X_i)_{i=1}^n \in \mathbb{A}_n^{\uparrow}(X)} \sum_{i=1}^n U_i(X_i). \tag{4.6}$$

Proof. By Lemma 6.1 in [49], we know that the result is true in the case $n = 2$, even considering choice criterions satisfying Assumption 3.20 instead of monetary utility functionals (the property of monotonicity doesn't take part in the proof). Now we proceed by induction to show that it holds for any number $n \in \mathbb{N}$ of agents. The basis of the induction is $n = 2$, and the hypothesis is that (4.6) holds for a number $m \in \mathbb{N}$ of agents. We have to prove that it is still true for $m + 1$ agents. Consider functionals U_i, $i = 1, \ldots, m + 1$, satisfying Assumption 3.20, and let $X \in L^\infty$. We have

$$\begin{aligned}
U_1 \square \cdots \square U_{m+1}(X) &\equiv \sup_{(X_i)_{i=1}^{m+1} \in \mathbb{A}_{m+1}(X)} \sum_{i=1}^{m+1} U_i(X_i) \\
&= \sup_{\zeta \in L^\infty} \Big\{ \sup_{(X_i)_{i=1}^m \in \mathbb{A}_m(\zeta)} \sum_{i=1}^m U_i(X_i) + U_{m+1}(X - \zeta) \Big\} \\
&= \sup_{(\zeta, (X-\zeta)) \nearrow X} \Big\{ \sup_{(X_i)_{i=1}^m \in \mathbb{A}_m(\zeta)} \sum_{i=1}^m U_i(X_i) + U_{m+1}(X - \zeta) \Big\} \\
&= \sup_{(\zeta, (X-\zeta)) \nearrow X} \Big\{ \sup_{(X_i)_{i=1}^m \in \mathbb{A}_m^{\uparrow}(\zeta)} \sum_{i=1}^m U_i(X_i) + U_{m+1}(X - \zeta) \Big\} \\
&= \sup_{(X_i)_{i=1}^{m+1} \in \mathbb{A}_{m+1}^{\uparrow}(X)} \sum_{i=1}^{m+1} U_i(X_i),
\end{aligned}$$

using both the basis and the hypothesis of the induction. The last equality is true by the fact that, if $X_i \nearrow \zeta$ and $\zeta \nearrow X$, then obviously $X_i \nearrow X$. This completes the inductive procedure, showing that equality (4.6) is true for any number n of economic agents.

□

4.1.2 Existence of Pareto Optimal Allocations

In Theorem 4.11 we give sufficient conditions to ensure that the set of Pareto optimal allocations is not empty, a fact that is not true in general (see [49, Section 6.3] for a counterexample). Note that, in view of Theorem 4.20, this also guarantees the existence of optimal risk sharing rules (Definition 4.19). The unique extra-property that we need on our choice criterions is the invariance in law, which is usually very simple to check.

Theorem 4.11. *Let U_i, $i = 1, \dots, n,$ be law-invariant choice functionals satisfying Assumption 3.20. Then, for any $X \in L^\infty$, the set of Pareto optimal allocations in $\mathbb{A}_n^\uparrow(X)$ is non-empty.*

Proof. Following, step by step, the same arguments used in [49], we consider a maximizing sequence $\{(X_1^k, \dots, X_n^k)\}_{k \in \mathbb{N}}$ of the sup-convolution problem (4.1), which can be chosen in $\mathbb{A}_n^\uparrow(X)$ by Lemma 4.10. Hence we have $X_i^k = \varphi_i^k(X)$ for some non-decreasing functions $\varphi_i^k : [\underline{x}, \overline{x}] \to \mathbb{R}$, for each $i = 1, \dots, n$ and $k \in \mathbb{N}$, where $[\underline{x}, \overline{x}] := [\operatorname{ess\,inf} X, \operatorname{ess\,sup} X]$. By possibly adding some constants to X_i^k, we may assume essinf $X_i^k = 0$ for any $i = 1, \dots, n-1$ and $k \in \mathbb{N}$. Since $\sum_{i=1}^n X_i^k = X$, it follows that $0 \le X_i^k \le \overline{x} - \underline{x}$, $\forall i = 1, \dots, n-1,$ and $\underline{x} \le X_n^k \le \overline{x}$. Therefore, for any $i = 1, \dots, n-1$ and $k \in \mathbb{N}$,

$$\varphi_i^k \in \mathcal{B} := \{f : [\underline{x}, \overline{x}] \to \mathbb{R} \, : \, |f| \le (\overline{x} - \underline{x}) \text{ and } f, (Id - f) \text{ are non-decreasing}\}.$$

Note that $\mathcal{B}$ is composed of 1-Lipschitz continuous functions, thus it is a bounded, closed, equicontinuous family. The Ascoli-Arzelà theorem ensures that, by possibly passing to a subsequence, φ_i^k converge to some $\varphi_i \in \mathcal{B}$ as $k \to \infty$, for each $i = 1, \dots, n-1$, uniformly on $[\underline{x}, \overline{x}]$. This implies

$$\varphi_n^k = Id - \sum_{i=1}^{n-1} \varphi_i^k \to Id - \sum_{i=1}^{n-1} \varphi_i =: \varphi_n \quad \text{as } k \to \infty$$

uniformly on $[\underline{x}, \overline{x}]$. Now we can check that $(\varphi_1(X), \dots, \varphi_n(X)) \in \mathbb{A}_n^\uparrow(X)$ is the maximizer we are looking for. Indeed, we have $U_1 \square \cdots \square U_n(X) = \lim_{k \to \infty} \sum_{i=1}^n U_i(X_i^k) = \sum_{i=1}^n U_i(\lim_{k \to \infty} \varphi_i^k(X)) = \sum_{i=1}^n U_i(\varphi_i(X))$, by assumption of L^∞-continuity of U_i, for any $i = 1, \dots, n$.

□

Remark 4.12. *Note that the existence of POAs implies, in particular, that the functional U defined in (4.2) is proper, with $dom(U) = L^\infty$ and $\bigcap_{i=1}^n dom(V_i) \neq \emptyset$.*

We can now complete the proof of Lemma 4.5.

Proof. [Lemma 4.5-(iii),(iv)] Let $(U_i)_{i=1}^n$ be law-invariant functionals satisfying property (S). In order to prove statement (iii), we fix $X \in L^\infty$ and $A \in \mathcal{F}$ s.t. $\mathbb{P}(A) > 0$ and X is not constant on A. By Theorem 4.11 we know that there exists a POA $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$, so that $U(X) = \sum_{i=1}^n U_i(X_i)$. The fact that X is not constant on A implies that, for some $j \in \{1, \dots, n\}$, X_j is not constant on A as well. By property (S) we have

$$\begin{aligned} U(X) &= \sum_{i=1}^n U_i(X_i) < \sum_{i=1}^n U_i(X_i \mathbf{1}_{A^c} + \mathbf{E}[X_i|A]\mathbf{1}_A) \\ &\leq U(X\mathbf{1}_{A^c} + \mathbf{E}[X|A]\mathbf{1}_A), \end{aligned}$$

which means that U is strictly risk-averse conditionally on any event and proves (iii). Now let $(U_i)_{i=1}^n$ be law-invariant functionals, with U_j strictly monotone for some $j \in \{1, \dots, n\}$. In order to prove statement (iv), we fix $X, Y \in L^\infty$ such that $X \leq Y$ and $\mathbb{P}(X < Y) > 0$. Consider a POA $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$, which exists by Theorem 4.11, and the allocation $(Y_1, \dots, Y_n)$ of Y given by:

$$Y_i = \begin{cases} X_j + (Y - X), & \text{if } i = j, \\ X_i, & \text{if } i \neq j. \end{cases}$$

This produces

$$U(X) = \sum_{i=1}^n U_i(X_i) < \sum_{i=1}^n U_i(Y_i) \leq U(Y),$$

which implies that U is strictly monotone and concludes the proof of Lemma 4.5. □

In the next chapter we strongly use a result proved in [49] for the case of two monetary utility functionals, which can be easily generalized to a more general context:

Lemma 4.13. *Let $U_1, \dots, U_n$ be choice functionals satisfying Assumption 3.20 and Assumption 4.3. Then the following statements are equivalent:*

(i) $\forall X \in L^\infty$, there exists a POA $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$,

(ii) $\forall \mu \in (L^\infty)^*,\ \partial V(\mu) \equiv \partial\Big(\sum_{i=1}^n V_i\Big)(\mu) = \sum_{i=1}^n \partial V_i(\mu)$.

Note how the assumption of law-invariance for all choice criterions $(U_i)_{i=1}^n$ ensures the existence of Pareto optimal allocations, by Theorem 4.11, and therefore (ii) also holds true.

Proof. $(i) \Rightarrow (ii)$: Fix any measure $\mu \in (L^\infty)^*$. The inclusion $\sum_i \partial V_i(\mu) \subseteq \partial(\sum_i V_i)(\mu)$ being always true, we must show the opposite one. If $\partial V(\mu) = \emptyset$, there is nothing to prove. So, let's assume $\partial V(\mu) \neq \emptyset$ and consider $X \in -\partial V(\mu)$, which is equivalent to say $\mu \in \partial U(X)$. From (i) there exists $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$ Pareto optimal and Proposition 4.9 applies, giving $\mu \in \partial U_i(X_i)$ for all $i = 1, \ldots, n$, that is $X_i \in -\partial V_i(\mu)$. This fact implies

$$X = \sum_{i=1}^n X_i \in -\sum_{i=1}^n \partial V_i(\mu),$$

thus concluding the first part of the proof.
$(ii) \Rightarrow (i)$: We have seen that Assumption 4.3 implies $\partial U(X) \neq \emptyset$ for any $X \in L^\infty$. Therefore, there exists some measure μ in $\partial U(X)$, that is

$$X \in -\partial V(\mu) = -\sum_{i=1}^n \partial V_i(\mu).$$

Therefore, there exists an allocation $(X_i)_i$ of X such that $X_i \in -\partial V_i(\mu)$ for all $i = 1, \ldots, n$, and Theorem 4.8 ensures that it is Pareto optimal. □

4.1.3 Some Examples of Sup-Convolution Problems

For the sake of completeness, we now state the natural extension to the case of any number n of agents, for some of the results given in [5] and [49] in the case of two agents, when dealing with functionals satisfying particular properties. For example, we can consider an aggregate of agents all working with the same kind of preferences, in the sense of the following definition.

Definition 4.14. *Given n agents endowed with choice criterions $U_1, \ldots, U_n$, we say that they have dilated utility measures (compare [5]) if, for any $i = 1, \ldots, n$, we can write*

$$U_i(X) = \alpha_i \bar{U}\Big(\frac{X}{\alpha_i}\Big), \quad \forall X \in L^\infty, \tag{4.7}$$

for some parameters $\alpha_i > 0, i = 1, \ldots, n$, and some functional $\bar{U}$ fulfilling Assumption 3.20.

Such agents have preference functionals of the same type, except for the *risk-tolerance coefficient* α_i, and this leads to the following result:

Proposition 4.15. *Let $(U_i)_{i=1}^n$ be dilated utility measures. Then, for any aggregate risk $X \in L^\infty$, we have*

$$U_1 \square \cdots \square U_n(X) = \alpha \bar{U}\left(\frac{X}{\alpha}\right), \quad \textit{where } \alpha = \sum_{i=1}^{n} \alpha_i. \tag{4.8}$$

In particular, we obtain that $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ given by

$$X_i = \frac{\alpha_i}{\alpha} X, \text{ for all } i = 1, \ldots, n,$$

is a Pareto optimal allocation, corresponding to the linear quota-sharing of X proportional to the risk-tolerance coefficients. As we have said, we know that this result is true in the particular case $n = 2$ (see [5], where the monotonicity property does not intervene). From this fact, and by the induction method, it clearly follows that relation (4.8) holds true for any number n of agents, calling to mind that the sup-convolution operator is associative.

Corollary 4.16. *Given an aggregate of n agents, which are all endowed with the same choice functional U, then*

$$U \square \cdots \square U(X) = nU\left(\frac{X}{n}\right), \quad \forall X \in L^\infty,$$

and the allocation that gives the same share $\frac{X}{n}$ of the total risk X to any agent, is Pareto optimal.

Moreover, if functional U in Corollary 4.16 is positively homogeneous, then

$$U \square \cdots \square U(X) = U(X), \quad \forall X \in L^\infty,$$

and any proportional sharing

$$(\beta_1 X, \ldots, \beta_n X), \quad \text{with } \beta_i \geq 0 \text{ and } \sum_{i=1}^{n} \beta_i = 1,$$

is Pareto optimal.

Now we introduce the following notion, given in [49], which allows us to formulate the proposition below.

Definition 4.17. *Let f, g be functions belonging to $\mathcal{D}$ (see Section 3.1) such that $f \leq g$. A non-decreasing function q on $[0, 1]$ is called flat on $\{f < g\}$ if it is almost surely constant on $\{f < g\}$ and $(q(0+) - q(0))(g - f)(0+) = 0$.*

Proposition 4.18. *Let $(U_i)_{i=1}^n$ be law-invariant, comonotone monetary utility functionals, with associated concave functions $f_i \in \mathcal{D}$ as in Theorem 3.8. Then*

(i) $U := U_1 \square \cdots \square U_n$ *is a law-invariant, comonotone m.u.f., with associated concave function* $f := \bigwedge_{i=1}^{n} f_i \in \mathcal{D}$*;*

(ii) $(X_i)_{i=1}^n \in \mathbb{A}_n^{\uparrow}(X)$ *is a POA if and only if* q_{X_i} *is flat on* $\{f_i > f\} \cap \{dq_X > 0\}$.

Once again, the fact that this result holds in the case $n = 2$ ([49]) makes the proof easy in the general case of any number n of agents. Therefore, we present just a sketch of it based, as before, on the induction method: assuming the result to hold for m agents, we have to prove that it remains true for $m+1$ agents. In this way statement (i) readily follows, whereas with regard to statement (ii) we note that, given an allocation $(X_i)_{i=1}^{m+1} \in \mathbb{A}_{m+1}^{\uparrow}(X)$ which is Pareto optimal for $(U_1, \ldots, U_m, U_{m+1})$, then the allocation $(X_i)_{i=1}^m \in \mathbb{A}_m^{\uparrow}(X - X_{m+1})$ is Pareto optimal for $(U_1, \ldots, U_m)$. We also recall that, for X_i comonotone, the equality $q_{(X_1+\cdots+X_m)} = q_{X_1} + \cdots + q_{X_m}$ holds, so that $q_{(X_1+\cdots+X_m)}$ is flat if and only if q_{X_i} is flat for any $i = 1, \ldots, m$.

4.2 Constraints on the Sup-Convolution Problem

As already pointed out, the solutions to the maximization problem (4.1), i.e., the Pareto optimal allocations, are jointly optimal for the agents, but do not pertain to the satisfaction of each agent individually. Indeed, it may occur that some agent worsens her position by passing from the initial risk endowment to a new one suggested by a POA. At this point, since we want each agent to be satisfied by the redistribution of the aggregate risk, we introduce the following notions:

Definition 4.19 ([49]). *Consider n agents endowed with choice functionals $U_1, \ldots, U_n$ and initial risky positions $(\xi_1, \ldots, \xi_n) \in L^\infty \times \cdots \times L^\infty$. Let $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ be an allocation of the total risk $X = \sum_{i=1}^n \xi_i$. Then we say that*

(i) $(X_1, \ldots, X_n)$ *satisfies the Individual Rationality constraints, if*

(IR) $\qquad U_i(X_i) \geq U_i(\xi_i), \;$ *for all* $\; i = 1, \ldots, n$*;*

(ii) $(X_1, \dots, X_n)$ *is an Optimal Risk Sharing (ORS) rule, if it satisfies both the Pareto optimality and the individual rationality constraints.*

Condition IR seems to be the most natural request in order to have all agents agree on how to modify their initial risk endowment, since in this way anyone has an incentive to enter into the contract, and this leads to the following theorem (see [49] for the case of two agents endowed with monetary utility functionals).

Theorem 4.20. *Consider* n *agents characterized by choice functionals* $(U_i)_{i=1}^n$ *satisfying Assumption 3.20, and initial endowments* $(\xi_1, \dots, \xi_n)$ *of the total risk* X*. Let* $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ *be a POA and* $p_i := U_i(X_i) - U_i(\xi_i)$, $i = 1, \dots, n$*, the utility increments experienced by the agents. Then the following statements hold:*

(i) $\sum_{i=1}^n p_i \geq 0$*;*

(ii) let $\pi_1, \dots, \pi_n$ *be constants s.t.* $\sum_{i=1}^n \pi_i = 0$*, then allocation* $(X_1 - \pi_1, \dots, X_n - \pi_n)$ *is an ORS rule if and only if* $\pi_i \leq p_i, \forall i = 1, \dots, n$*.*

Proof. From the Pareto optimality we have

$$\sum_{i=1}^n p_i = \sum_{i=1}^n U_i(X_i) - \sum_{i=1}^n U_i(\xi_i) = \sup_{(\zeta_i)_{i=1}^n \in \mathbb{A}_n(X)} \sum_{i=1}^n U_i(\zeta_i) - \sum_{i=1}^n U_i(\xi_i) \geq 0,$$

which proves statement (i). Let us now show (ii). For any choice of constants $(\pi_i)_{i=1}^n$ summing up to zero, $(X_1 - \pi_1, \dots, X_n - \pi_n)$ is a POA of X, as seen in Remark 4.7. Therefore, by definition, it is an ORS rule if and only if $U_i(X_i - \pi_i) \geq U_i(\xi_i)$, $\forall i = 1, \dots, n$, which is equivalent to have

$$\pi_i \leq U_i(X_i) - U_i(\xi_i) = p_i, \quad \forall i = 1, \dots, n,$$

from the cash-invariance property. □

This theorem ensures the existence of optimal risk sharing rules of a given aggregate risk, provided the existence of a Pareto optimal allocation. We actually get more than this. To any PAO we have associated a set of prices $(\pi_i)_{i=1}^n$ which make the IR constraints satisfied. We call $p_1, \dots, p_n$ *indifference prices*, or indifference pricing rules, since agents are indifferent, from their utility point of view, to either carrying out this transaction at these prices or not carrying it out at all. We note that price p_i corresponds to the maximal amount agent i is willing to pay to enter into the contract, i.e., to change her position from the initial one ξ_i to the new one X_i.

Remark 4.21. *If the initial risk endowment is already optimal in the sense of Pareto, then for any other POA we consider, we have a unique vector of prices making it an ORS rule. Indeed, using the notation of Theorem 4.20, we get* $\sum_{i=1}^n p_i = 0$ *and the only possibility to price the contract is via the indifference prices, i.e.* $\pi_i = p_i$, $\forall i = 1, \dots, n$. *Since in this case each agent maintains her own level of satisfaction, then it is convenient to do nothing. Otherwise, if the initial endowment does not constitute a solution to the sup-convolution problem, then any POA admits an infinite set of suitable prices, which form the polyhedral space* $\{(\pi_i)_{i=1}^n : \sum_{i=1}^n \pi_i = 0,\ \pi_i \le p_i\}$. *At this point, it is the market power of the economic agents involved in the problem that determines the unique vector of prices of an optimal contract.*

For example, if there is an agent, say i, with the power to decide the price of the contract for any agent (ultimatum game), then the problem to solve is the maximization of her utility, under the individual rationality constraints for the other agents:

$$(P_i) \qquad \begin{cases} \sup U_i\big(X - \sum_{j\neq i}(X_j - \pi_j)\big) \\ U_j(X_j - \pi_j) \ge U_j(\xi_j),\ j \neq i. \end{cases}$$

In solving problem (P_i), we obtain ORS rules $(X_1 - \pi_1, \dots, X_n - \pi_n)$, where obviously X_i and π_i are defined by $X_i = X - \sum_{j\neq i} X_j$ and $\pi_i = -\sum_{j\neq i}\pi_j$. In particular, we get all the POAs $(X_1, \dots, X_n)$ with associated prices uniquely determined by $\pi_j = U_j(X_j) - U_j(\xi_j) = p_j$, $\forall j \neq i$. This means that for each agent $j \neq i$ we consider her indifference price p_j, so that for agent i we have her best possible (i.e. minimal) price $\pi_i = -\sum_{j\neq i} p_j$. Therefore, agent i achieves the maximum utility that she is eligible for in an ORS rule, given by

$$U_i(X_i - \pi_i) = U(X) - \sum_{j\neq i} U_j(\xi_j).$$

Remark 4.22. *In an ultimatum game this is the rational solution suggested by the theory, that is, for the proposer to offer the biggest possible prices* $(p_j)_{j\neq i}$ *(or rather* $(p_j - \epsilon_j)_{j\neq i}$ *for some small quantities* $\epsilon_j > 0$ *s.t.* $-\sum_{j\neq i}(p_j - \epsilon_j) \le p_i$*), and for the responders to accept* *them. In spite of that, we empirically have a different situation since, most frequently, we observe fair prices (see [13], [45] and [64] for a justification).*

4.3 Monotone Approximations in the ORS Problem

So far, in this chapter, no distinction is made between choice functionals which do or do not satisfy monotonicity. In this section, on the contrary, we just emphasize if the criterions involved in the ORS problem have or haven't got this property. In line with the notation of the preceding chapter, for any non-monotone criterion $U : L^\infty \to \mathbb{R}$ we denote by U^m its best monotone version (3.49), which in particular is a monetary utility functional.

Let us start with a preparatory result, which will allow us to make a comparison between problems involving non-monotone criterions, and the associated ones which only involve monetary utility functionals.

Lemma 4.23. *Let $U_1, \dots, U_n$ be choice functionals satisfying Assumption 3.20 and Assumption 4.3, and let at least one of these be monotone. Then, for any POA $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$,*

$$X_i \in M(U_i), \ \forall i = 1, \dots, n, \tag{4.9}$$

where $M(U_i)$ is the domain of monotonicity of U_i (see Definition 3.30).

This means that, in an optimal redistribution of the total risk, each agent will always take upon herself a risky position which lies in the set where she behaves monotonically.

Proof. Let $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ be a POA of the total risk X. Theorem 4.8 ensures the existence of a measure $\mu \in (L^\infty)^*$ such that $\mu \in \partial U_i(X_i)$, for any $i = 1, \dots, n$. By hypothesis there exists an agent, say j for some $j \in \{1, \dots, n\}$, which satisfies the axiom of monotonicity. The argument that follows (3.2) implies $\partial U_j(X_j) \subseteq \mathrm{dom}(V_j) \subseteq (L^\infty)^*_+$, so that μ is a positive measure and (4.9) readily follows by (3.48).

□

For a collection of functionals $U_1, \dots, U_n$ on $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, we denote by (P) and (P^m) respectively, the sup-convolution problems of the original criterions and of their best monotone approximations:

$$(P) \qquad U_1 \square \cdots \square U_n(X) = \sup_{(X_1,\dots,X_n)\in\mathbb{A}_n(X)} \sum_{i=1}^{n} U_i(X_i),$$

$$(P^m) \qquad U_1^m \square \cdots \square U_n^m(X) = \sup_{(X_1,\dots,X_n)\in\mathbb{A}_n(X)} \sum_{i=1}^{n} U_i^m(X_i).$$

Moreover, we call U and $\widetilde{U}$ the concave functionals on $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ obtained from these problems:

$$U(X) := U_1 \square \cdots \square U_n(X), \qquad \widetilde{U}(X) := U_1^m \square \cdots \square U_n^m(X) \tag{4.10}$$

and V, $\widetilde{V}$ their convex conjugate functions defined on $L^\infty(\Omega, \mathcal{F}, \mathbb{P})^*$.

Theorem 4.24. *Let $U_1, \dots, U_n$ be as in Lemma 4.23. Then U and $\widetilde{U}$ introduced in (4.10) describe the same monetary utility functional on $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$:*

$$U(X) = \widetilde{U}(X), \ \ \forall X \in L^\infty. \tag{4.11}$$

Proof. The fact that U and $\widetilde{U}$ are monetary utility functionals is an immediate consequence of Lemma 4.5 and the arguments that precede it, which also imply that (U, V), as well as $(\widetilde{U}, \widetilde{V})$, are $\langle L^\infty, (L^\infty)^* \rangle-$ conjugate. From relation (4.3) we have

$$V(\mu) = \sum_{i=1}^{n} V_i(\mu), \quad \text{where} \quad \operatorname{dom}(V) = \bigcap_{i=1}^{n} \operatorname{dom}(V_i) \subseteq (L^\infty)^*_+,$$

since $\operatorname{dom}(V_j) \subseteq (L^\infty)^*_+$ for some $j \in \{1, \dots, n\}$. On the other hand, by (3.51) we obtain

$$\widetilde{V}(\mu) = \sum_{i=1}^{n} V_i^m(\mu) = \begin{cases} \sum_{i=1}^{n} V_i(\mu), & \text{on } \bigcap_{i=1}^{n} \operatorname{dom}(V_i) \bigcap (L^\infty)^*_+, \\ +\infty, & \text{elsewhere,} \end{cases}$$

so that $V = \widetilde{V}$ on $(L^\infty)^*$, and therefore $U = \widetilde{U}$ on L^∞.

□

Equality (4.11) means that the consideration of criterions $(U_1, \dots, U_n)$ or $(U_1^m, \dots, U_n^m)$ leads, for any aggregate risk, to the same maximal overall utility, although it does not say anything about which allocations realize or approximate this supremum. Let us now take into account exactly such allocations, that is, how the total risk can be optimally re-shared among the involved agents, which is the subject matter of this chapter. The following result is just a first answer in this direction.

Corollary 4.25. *Let $U_1, \dots, U_n$ be as in Lemma 4.23. Then, for any aggregate risk $X \in L^\infty$, any solution to problem (P) is also a solution to problem (P^m).*

By Theorem 4.8 this means that, for each $X \in L^\infty$, the following relation between the sets of Pareto optimal allocations holds true:

$$\{\text{POAs for } (U_1, \dots, U_n)\} \subseteq \{\text{POAs for } (U_1^m, \dots, U_n^m)\}.$$

Therefore, provided that some agent has monotone preferences, for any other agent involved in the redistribution of the risk it does not matter if she behaves monotonically or not. Indeed, the optimal allocations we find by solving the problem w.r.t. the original criterions $(U_i)_{i=1}^n$ are also optimal w.r.t. their adjusted versions $(U_i^m)_{i=1}^n$.

Proof. If problem (P) admits no solutions, then there is nothing to prove. So, let's assume there exists an allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ which is Pareto optimal w.r.t. the choice functionals $(U_i)_{i=1}^n$. Lemma 4.23 implies $X_i \in M(U_i)$ for any $i = 1, \dots, n$, thus we have $U_i(X_i) = U_i^m(X_i)$ by Theorem 3.31. At this point, Theorem 4.24 gives us

$$\widetilde{U}(X) = U(X) = \sum_{i=1}^n U_i(X_i) = \sum_{i=1}^n U_i^m(X_i),$$

which makes $(X_1, \dots, X_n)$ Pareto optimal w.r.t. the monotone functionals $(U_i^m)_{i=1}^n$ as well. □

Corollary 4.26. *Let $U_1, \dots, U_n$ be as in Lemma 4.23, and let $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ be a solution of both problems (P) and (P^m). Then, any vector of prices that agents characterized by U_i^m's are willing to pay for this contract, is also optimal for agents characterized by U_i's.*

Proof. Theorem 4.20 is the recipe to find all ORS rules associated to a given Pareto optimal allocation. It only uses increments of utility that agents experience by passing from the initial endowment $(\xi_1, \dots, \xi_n)$ of the total risk X, to such a POA. We assume the same notation for the utility increments w.r.t. the original choice functionals $U_1, \dots, U_n$:

$$p_i := U_i(X_i) - U_i(\xi_i), \ \forall i = 1, \dots, n,$$

and denote as follows the relative set of acceptable prices:

$$\Pi := \{(\pi_i)_{i=1}^n \in \mathbb{R}^n : \sum_{i=1}^n \pi_i = 0, \ \pi_i \le p_i, \forall i = 1, \dots, n\}.$$

In the same way we define

$$p_i^m := U_i^m(X_i) - U_i^m(\xi_i), \ \forall i = 1, \dots, n,$$

as the utility increments w.r.t. the monotone adjusted versions $(U_i^m)_{i=1}^n$, and

$$\Pi^m := \{(\pi_i^m)_{i=1}^n \in \mathbb{R}^n : \sum_{i=1}^n \pi_i^m = 0, \ \pi_i^m \leq p_i^m, \forall i = 1, \dots, n\}$$

as the relative set of acceptable prices. With this notation, the statement we are going to prove can be formulated as $\Pi^m \subseteq \Pi$. Whereas we have seen that agents with choice criterions U_i and U_i^m give the same value to the optimal share X_i, this is no longer true with regard to the initial risk share ξ_i. Indeed it may happen that $\xi_i \notin M(U_i)$ and in this case $U_i^m(\xi_i) > U_i(\xi_i)$. Therefore, in general, we can only say that $U_i^m(\xi_i) \geq U_i(\xi_i)$, which gives the following relation between the indifference prices:

$$p_i^m \leq p_i. \tag{4.12}$$

Hence, for any ORS rule $(X_i - \pi_i^m)_{i=1}^n$ w.r.t. the functionals U_i^m's, that is, for any $(\pi_i^m)_{i=1}^n \in \Pi^m$, we have $\pi_i^m \leq p_i^m \leq p_i$, which yields $(\pi_i^m)_{i=1}^n \in \Pi$ and makes $(X_i - \pi_i^m)_{i=1}^n$ an ORS rule w.r.t. U_i's as well.

□

Let us now focus our attention on non-monotone choice functionals of "mean-variance" type (3.38). In this case we can state more interesting results, by relying on Proposition 3.32 which, for any mean-variance functional U_δ^{mv}, yields the characterization of the relative monotone-mean-variance functional U_δ^{mmv} and ensures, $\forall\, X \in L^\infty$, the existence of a payoff $Y \in L^\infty$ such that $Y \leq X$ and $U_\delta^{mv}(Y) = U_\delta^{mmv}(X)$.

Theorem 4.27. *Let $U_1, \dots, U_n$ be functionals satisfying Assumption 3.20 and Assumption 4.3, such that at least one is strictly monotone, and the non-monotone ones are of type (3.38). Then, for any aggregate risk $X \in L^\infty$, problems (P) and (P^m) admit the same set of solutions, that is*

$$\{POAs \ \ for \ (U_1, \dots, U_n)\} = \{POAs \ \ for \ (U_1^m, \dots, U_n^m)\}. \tag{4.13}$$

Proof. The inclusion in one sense being immediate by Corollary 4.25, let us prove the other inclusion in (4.13). In order to do this, let's assume $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ to solve problem (P^m). If $X_i \in M(U_i)$ for any $i = 1, \ldots, n$, it is easy to see that $(X_1, \ldots, X_n)$ solves problem (P) as well, by the same argument used to prove Corollary 4.25. Now suppose $X_j \notin M(U_j)$ for some $j \in \{1, \ldots, n\}$, which in particular implies that U_j is non-monotone, hence a mean-variance functional by assumption. We know that, by hypothesis, there is an agent, say k, with strictly monotone preferences, where obviously $k \in \{1, \ldots, n\} \setminus \{j\}$. Since $X_j \notin M(U_j)$, Theorem 3.31 implies $U_j^m(X_j) > U_j(X_j)$, and Proposition 3.32 ensures that the supremum in (3.50) is actually a maximum for U_j. Therefore, there exists a payoff $Y \in L^\infty$ such that $Y < X_j$ and $U_j^m(Y) = U_j(Y) = U_j^m(X_j)$. Let us consider the allocation $(\zeta_1, \ldots, \zeta_n) \in \mathbb{A}_n(X)$ given by

$$\zeta_i = \begin{cases} Y, & \text{if } i = j, \\ X_k + (X_j - Y), & \text{if } i = k, \\ X_i, & \forall\, i \in \{1, \ldots, n\} \setminus \{j, k\}. \end{cases}$$

Strict monotonicity of U_k implies $U_k^m(\zeta_k) = U_k(\zeta_k) > U_k(X_k) = U_k^m(X_k)$, so that

$$\sum_{i=1}^n U_i^m(\zeta_i) > \sum_{i=1}^n U_i^m(X_i) = U_1^m \Box \cdots \Box U_n^m(X),$$

which clearly contradicts the definition of sup-convolution. □

This is interesting from an economic point of view: whereas in Chapter 3 we have seen how the lack of monotonicity may lead to pathological situations, here we have that the optimal risk sharing does not take into account the fact that some (but not all!) choice criterions may fail this property. On the other hand, whereas the Pareto optimal redistribution of the total risk is not affected by the possible lack of monotonicity by some agents, this is no longer true for the price of the contract. Indeed, by imposing the individual rationality constraints and looking for ORS rules, we have only the inclusion in one sense among the sets of optimal solutions, as stated in the following corollary:

Corollary 4.28. *Let $U_1, \ldots, U_n$ be as in Theorem 4.27. Then we have the following relation between the solutions to the ORS problem w.r.t. the functionals U_i's and the solutions to the ORS problem w.r.t. the functionals U_i^m's:*

$$\{ORS \text{ rules for } (U_1^m, \ldots, U_n^m)\} \subseteq \{ORS \text{ rules for } (U_1, \ldots, U_n)\}. \tag{4.14}$$

Proof. It readily follows from Theorem 4.27 and Corollary 4.26. □

From the proof of Corollary 4.26 it becomes clear that, in general, the maximal price that a non-monotone agent is willing to pay to enter into a contract is higher than that accepted by her monotone approximation, thus making the inverse inclusion in (4.14) untrue. On the other hand, if the initial risk endowment $\xi_i \in M(U_i)$ for any $i = 1, \ldots, n$, then the equality holds in (4.12) for any agent, yielding the same ORS rules for the agents characterized by $(U_1, \ldots, U_n)$ and for those characterized by $(U_1^m, \ldots, U_n^m)$, which implies the coincidence of the sets in (4.14).

Example 4.29. *Theorem 4.27 applies, e.g., to the problem of sharing an aggregate risk between a mean-variance agent and an agent with entropic utility (see Proposition 5.1 and compare also Proposition 5.7) or between a mean-variance agent and an agent with semi-deviation utility with $p \neq +\infty$ (see Subsection 5.1.4).*

Remark 4.30. *We obtain a result as that stated in Theorem 4.27, also when considering an entropic and a standard-deviation choice criterion, since the strict concavity of the first one and the strict convexity of its dual transform imply the existence of a unique (up to a constant) solution to both problems (P) and (P^m).*

Chapter 5

Explicit Characterization of Optimal Risk Sharing Rules

In this chapter we formulate and solve some specific problems of optimal risk sharing. We consider agents characterized by preference functionals as those introduced in the previous chapters and provide optimal rules to share, among them, a generic aggregate risk $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Recall that the choice criterions studied in Section 3.2 and in Section 3.4 satisfy law-invariance and Lebesgue property, thus allowing us to work with measures in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and to apply other important results. Here we do not consider the initial risk endowment of the agents, hence looking for Pareto optimal allocations without interest in the prices of the contracts. This is motivated by the fact that, once we have a POA, only simple calculations are required to find the suitable prices, as shown in Theorem 4.20.

5.1 Optimal Risk Sharing: the Case of Two Agents

In any case contemplated in this section we consider two agents, denoted by $i = 1, 2$, with preferences modelled by some law-invariant choice functionals, say U_1 and U_2, satisfying Assumption 3.20 (Assumption 4.3 is automatically satisfied). Denote by V_1 and V_2 the respective convex conjugate functions, defined on the dual space $L^\infty(\Omega, \mathcal{F}, \mathbb{P})^*$, and recall that, under the Lebesgue property, their effective domain is contained in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. Let

$U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be the concave functional solution to the sup-convolution problem:

$$U(X) := U_1 \square U_2(X), \quad \forall X \in L^\infty, \tag{5.1}$$

and $V : L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ its Fenchel-Legendre transform:

$$V(\mu) = V_1(\mu) + V_2(\mu), \quad \text{with} \quad \text{dom}(V) = \text{dom}(V_1) \cap \text{dom}(V_2) \neq \emptyset. \tag{5.2}$$

Now the law-invariance property ensures the existence of Pareto optimal allocations by Theorem 4.11, and the equality

$$\partial V = \partial V_1 + \partial V_2 \tag{5.3}$$

is also true by Lemma 4.13.

5.1.1 The Prudent Entropic-Agent

• `Entropic vs Mean-variance`.

Proposition 5.1. *Let U_1 be the entropic utility (3.17) with parameter $\gamma > 0$, and U_2 the mean-variance principle (3.38) with parameter $\delta > 0$. Then, for any aggregate risk $X \in L^\infty$, there exists a unique (up to a constant) Pareto optimal allocation $(X_1, X_2) \in \mathbb{A}_2^\uparrow(X)$. In particular, X_1 (resp. X_2) is, pointwise, a convex (resp. concave) function of the total risk.*

In general this means that, if an agent with entropic utility and an agent with mean-variance preferences optimally share an aggregate risk X, the former one especially takes the lowest risks (corresponding to the biggest values of X), whereas the latter especially takes the worst (corresponding to the smallest values of X).

Proof. Fix a risky position $X \in L^\infty$. We know that the set of Pareto optimal allocations is not empty by the law-invariance of U_1 and U_2. Let us prove that, up to a constant, it consists of a unique pair in $L^\infty \times L^\infty$. By Theorem 4.8, for any POA $(X_1, X_2) \in \mathbb{A}_2(X)$ there exists an element $Z \in$dom(V) such that $X_i \in -\partial V_i(Z)$, that is, $Z \in \partial U_i(X_i), \forall i = 1, 2$. In particular this implies $Z \in \partial U(X)$, for U defined in (5.1), by Proposition 4.9. On the other hand, V inherits strict convexity (on its domain) from V_1 and V_2, thus leading to a unique supergradient of U at X:

$$\partial U(X) = \{Z_X\}, \quad \text{for some } Z_X \in \text{dom}(V) = \mathcal{Z} \cap L^2,$$

by Theorem 2.9. Therefore, Theorem 3.16 and Theorem 3.28 yield

$$X \in -\partial V(Z_X) = \Big\{ -\gamma \ln Z_X - \frac{Z_X}{2\delta} + c : \; c \in \mathbb{R} \Big\}, \tag{5.4}$$

with

$$X_1 \in -\partial V_1(Z_X) = \{ -\gamma \ln Z_X + c : \; c \in \mathbb{R} \}$$

and

$$X_2 \in -\partial V_2(Z_X) = \Big\{ -\frac{Z_X}{2\delta} + c : \; c \in \mathbb{R} \Big\}.$$

Hence the uniqueness (up to a constant) of Pareto optimal allocations clearly follows and we can show the last assertion of the proposition. From (5.4) we have

$$X = -\gamma \ln Z_X - \frac{Z_X}{2\delta} + c_X,$$

for some constant c_X univocally determined by the condition $\mathbf{E}[Z_X] = 1$. Now the pointwise relation between X and Z_X can be written as

$$X = f(Z_X), \text{ meaning that } X(\omega) = f(Z_X(\omega)),$$

where $f : \mathbb{R}^+ \to \mathbb{R}$ is convex and decreasing. Since it is a one-to-one function, we can also write

$$Z_X = g(X),$$

again as a pointwise relation, with $g : \mathbb{R} \to \mathbb{R}^+$ convex and decreasing function. Therefore, pointwise, we have

$$X_1 = -\gamma \ln Z_X + c_1 = -\gamma \ln(g(X)) + c_1, \quad \text{for some } c_1 \in \mathbb{R},$$

convex and increasing function of X, and

$$X_2 = -\frac{Z_X}{2\delta} + c_2 = -\frac{g(X)}{2\delta} + c_2, \quad \text{for some } c_2 \in \mathbb{R}$$

concave and increasing function of X, as previously declared.

□

• `Entropic vs Standard-deviation`.

With the same reasoning as before, we can solve the sup-convolution problem for an entropic and a standard-deviation agent, obtaining similar results. This time U_2 is defined as in (3.39) and we get

$$X \in \{-\gamma \ln Z_X - cZ_X + d : \; c \in \mathbb{R}_0^+, d \in \mathbb{R}\},$$

for some $Z_X \in \mathcal{Z}$ (which exists unique), with

$$X_1 \in \{-\gamma \ln Z_X + c : \; c \in \mathbb{R}\}$$

and

$$X_2 \in \{-cZ_X + d : \; c \in \mathbb{R}_0^+, d \in \mathbb{R}\},$$

by Theorem 3.16 and Theorem 3.29. In this way we obtain the unique (up to a constant) POA $(X_1, X_2) \in \mathbb{A}_2^{\uparrow}(X)$, given by

$$X_1 = -\gamma \ln(g(X)) + c_1, \quad \text{for some } \; c_1 \in \mathbb{R},$$

and

$$X_2 = -c_2 g(X) + c_3, \quad \text{for some } \; c_2 \in \mathbb{R}_0^+ \text{ and } c_3 \in \mathbb{R},$$

where g is a convex and decreasing function. This means that, pointwise, X_1 is a convex and increasing function of the total risk X, whereas X_2, if not constant, is a concave and increasing function of X. It is clear that "X_2 constant" corresponds to the case of the entropic agent taking on all of the aggregate risk, a fact that, as we have seen, cannot occur in the optimal sharing between an entropic and a mean-variance agent.

• `Entropic vs Semi-deviation`.

Let agent 2 be characterized by the semi-deviation utility (3.22) with $p = 2$. Once again, for any $X \in L^\infty$ there exists a unique (up to a constant) POA $(X_1, X_2) \in \mathbb{A}_2^{\uparrow}(X)$ characterized by a unique $Z_X \in \cap_i \partial U_i(X_i)$. In particular, since $X \in -\partial V(Z_X) = -\sum_i \partial V_i(Z_X)$, Theorem 3.16 and Theorem 3.17 produce one of the following situations: either $\partial V_2(Z_X) = \{c : \; c \in \mathbb{R}\}$ and the risk X is totally charged to the entropic agent ($X_1 = X$), or $\partial V_2(Z_X)$ takes the form (3.30) and therefore

$$X \in -\partial V(Z_X) = \Big\{ -\gamma \ln Z_X - \frac{\mathbf{E}[Y]}{1 - z_X} Z_X + Y + c : \; c \in \mathbb{R}, \, Y \in L_+^\infty \text{ and } Y \mathbf{1}_{\{Z_X \neq z_X\}} \equiv 0 \Big\},$$

with $z_X = \min Z_X$. More precisely, in the latter case we have

$$X_1 \in -\partial V_1(Z_X) = \{-\gamma \ln Z_X + c : \, c \in \mathbb{R}\}$$

and

$$X_2 \in -\partial V_2(Z_X) = \Big\{ -\frac{\mathbf{E}[Y]}{1-z_X} Z_X + Y + c : \, c \in \mathbb{R}, \, Y \in L^\infty_+ \text{ and } Y \mathbf{1}_{\{Z_X \neq z_X\}} \equiv 0 \Big\}.$$

Lemma 3.7 ensures that $-Z_X$ and X are comonotone random variables, and therefore the total risk X takes its biggest values on the set $\{\omega : Z_X(\omega) = z_X\}$, which characterizes an interval of type $[\beta, \operatorname{ess\,sup} X]$. Clearly here X_1 is constant, so that the entire risk is charged to the semi-deviation agent. On the other hand, when X takes values in $[\operatorname{ess\,inf} X, \beta)$, for X, X_1 and X_2 we find the same behaviour as in the previous examples, that is, X_1 (resp. X_2) is a convex (resp. concave) increasing function of X. In particular, we have that the agent who especially assumes the worst risks is the semi-deviation one.

Remark 5.2. *The cases studied in this subsection show that an agent endowed with entropic utility is prudent towards extreme risks. Indeed, in all these situations we have seen that, in general, she especially takes the smaller risks, leaving the worse to the other agent, although it may happen that she takes the entire risk. The same situation occurs when considering the problem of sharing a risk between an entropic-agent and an $AV@R$-agent characterized by a functional of type (3.14). In this case (see [49]) the $AV@R$-agent entirely takes the worst risks and the resulting optimal sharing consists of a call option, written on the total risk and offered to the entropic-agent. In this way the risk charged to the entropic-agent results capped from below, thus producing a typical insurance contract (called stop-loss contract) and confirming the conservative nature of such an agent.*

5.1.2 AV@R-Agent vs Agent with Property (S)

Now we want to compare the $AV@R$-criterion defined in (3.14) with our most-representative non-monotone functionals. With the following proposition we provide a more general result which includes choice criterions (3.38) and (3.39) as particular cases.

Proposition 5.3. *Let U_1 be given by (3.14), and let U_2 be a law-invariant functional satisfying Assumption 3.20 and property (S) (see Definition 3.10). Then, for any aggregate* *risk $X \in L^\infty$, there exists a unique (up to a constant) POA in $\mathbb{A}_2^{\uparrow}(X)$, given by*

$$(X_1, X_2) := (-(X-l)^- + (X-u)^+, (l \vee X) \wedge u), \quad \textit{for some } l, u \in \mathbb{R}. \tag{5.5}$$

This means that the optimal sharing consists in the exchange of two European options written on the total risk X. Once again what we obtain is a typical insurance contract (called *limited stop-loss contract*), expressed by (5.5), where the insurer's risk share (X_2) has floor l and is capped at level u.

Proof. Let $(X_1, X_2) \in \mathbb{A}_2^{\uparrow}(X)$ be a POA of a given total risk $X \in L^\infty$. We want to show that it has shape (5.5). Theorem 4.8 ensures the existence of a density Z in $\partial U_1(X_1) \cap \partial U_2(X_2)$, and the reasoning that follows (3.16) implies that $0 \le Z \le 1/\lambda$, with

$$-\int_0^t q_{-Z}(s)ds \le f_\lambda \equiv \frac{t}{\lambda} \wedge 1, \ \forall t \in [0,1],$$

and X_1 constant on $\{Z \in \left(0, \frac{1}{\lambda}\right)\}$. On the other hand, Lemma 3.11 implies that X_2 is constant on the sets $\{Z = 0\}$ and $\{Z = 1/\lambda\}$. Now (Z, X_1), as well as (Z, X) and (Z, X_2), are anticomonotone random variables by Lemma 3.7, so that X_1 takes its biggest values on $\{Z = 0\}$ and the smallest ones on $\{Z = 1/\lambda\}$. Therefore, since X_1 and X_2 increase with X, we get

$$X_1 = -(X-l)^- + (X-u)^+ \text{ and } X_2 = (l \vee X) \wedge u, \tag{5.6}$$

for some thresholds l and u, as stated in (5.5).

At this point the uniqueness arises from the fact that the Pareto optimal allocations constitute a convex space in which each element has to assume this form. Then, fix any POA $(Y_1, Y_2) \in \mathbb{A}_2^{\uparrow}(X)$ and assume it is different from (X_1, X_2), in the sense that they do not differ only by a constant. From the first part of the proof, we know that it has the same shape as (X_1, X_2), so that it is characterized by a pair $(\hat{l}, \hat{u}) \neq (l, u)$. By convexity, for any $\alpha \in (0,1)$ the allocation given by $\xi_i = \alpha X_i + (1-\alpha)Y_i$, $i = 1, 2$, is Pareto optimal as well. On the other hand, since $(\hat{l}, \hat{u}) \neq (l, u)$, allocation (ξ_1, ξ_2) cannot have the desired shape, thus leading to a contradiction. □

Remark 5.4. *Consider now a convex combination of AV@R-criterions with parameters $\lambda_j \in (0,1], j = 1, \dots, m$, once again in convolution with a functional satisfying property (S).*

In this case, as optimal risk sharing rules we obtain allocations corresponding to a finite sum of European options written on the total risk.

Indeed, for any $\alpha_j \geq 0$ s.t. $\sum_{j=1}^{m} \alpha_j = 1$, $U_1 := \sum_{j=1}^{m} \alpha_j U_{\lambda_j}$ is a law-invariant comonotone m.u.f., characterized by the concave function $f_{U_1} := \sum_{j=1}^{m} \alpha_j f_{\lambda_j}$, where $(f_{\lambda_j})_{j=1}^{m}$ satisfy (3.16). By means of Lemma 3.9, for any POA $(X_1, X_2) \in \mathbb{A}_2(X)$ and any density $Z \in \partial U_1(X_1) \cap \partial U_2(X_2)$, we get

$$-\int_0^t q_{-Z}(s)ds \leq \sum_{j=1}^{m} \alpha_j f_{\lambda_j} = \sum_{j=1}^{m} \alpha_j \Big(\frac{t}{\lambda_j} \wedge 1\Big), \tag{5.7}$$

with q_{X_1} constant where this inequality is strict. On the other hand, when the equality holds in (5.7) then Z is constant, so that X_2 is also constant by Lemma 3.11. At this point, the announced form for the optimal sharing rules readily follows.

• `AV@R vs Mean-Variance/Standard-deviation`.

Both the mean-variance and the standard-deviation principles satisfy the conditions required in Proposition 5.3 on agent 2. This means that, for any risk X we consider, the optimal sharing between an $AV@R$-agent and a mean-variance (or standard-deviation) agent consists in the exchange of at the most two European options. In order to better understand this fact, consider the interval $[\operatorname{ess\,inf} X, \operatorname{ess\,sup} X]$ of the essential oscillations of X. Share it in three consecutive subintervals I_1, I_2, I_3 identified by

$$J_1 = \{Z = 1/\lambda\},\ J_2 = \{Z \in (0, 1/\lambda)\},\ J_3 = \{Z = 0\},\ \text{for some } Z \in \partial U(X),$$

in the sense that, almost surely, $\omega \in J_k$ iff $X(\omega) \in I_k$, $k = 1, 2, 3$. This produces the general form of the contract we found in Proposition 5.3, where the $AV@R$-agent assumes the risk whenever events occur in J_1 or J_3, whereas the second agent assumes the entire risk in J_2. However, it may happen that one or two of these intervals disappears, as shown in the following example.

Example 5.5. *Consider the particular case of Proposition 5.3 where agent 2 has mean-variance preferences with parameter $\delta > 0$. Let the aggregate risk X have essential oscillations bounded in the following way:*

$$(esssupX - essinfX) < \frac{1}{2\delta} \wedge \frac{1}{2\delta}\Big(\frac{1}{\lambda} - 1\Big). \tag{5.8}$$

Then we find $Z \in \cap_i \partial U_i(X_i)$, for some $(X_i)_i \in \mathbb{A}_n(X)$, such that $Z \in (0, 1/\lambda)$ $\mathbb{P}-$a.s.. In this case, by definition, intervals J_1 and J_3 disappear and the aggregate risk X is totally charged to the mean-variance agent.

To prove what is stated in Example 5.5, observe that (5.8) in particular produces

$$X - \mathbf{E}[X] \leq \operatorname{ess\,sup} X - \operatorname{ess\,inf} X < \frac{1}{2\delta}$$

and

$$X - \mathbf{E}[X] \geq -(\operatorname{ess\,sup} X - \operatorname{ess\,inf} X) > -\frac{1}{2\delta} \vee -\frac{1}{2\delta}\Big(\frac{1}{\lambda} - 1\Big) \geq -\frac{1}{2\delta}\Big(\frac{1}{\lambda} - 1\Big).$$

Therefore, we have

$$-\frac{1}{2\delta}\Big(\frac{1}{\lambda} - 1\Big) < X - \mathbf{E}[X] < \frac{1}{2\delta}, \tag{5.9}$$

which is equivalent to say

$$0 < 1 - 2\delta(X - \mathbf{E}[X]) < \frac{1}{\lambda}. \tag{5.10}$$

Put $\overline{Z} := 1 - 2\delta(X - \mathbf{E}[X]) \in (0, 1/\lambda)$ and observe that $\overline{Z} \in \partial U_1(0) \cap \partial U_2(X)$. This implies $(0, X)$ to be a POA, by Theorem 4.8, and in fact it is the unique one (up to a constant), by the previous proposition. This shape of the optimal re-sharing is not surprising if we consider the fact that mean-variance preferences only penalize the variance of financial positions. Therefore, when a payoff has a sufficiently small variability, a mean-variance agent associates a high level of satisfaction to it, thus making it favourable for her to take on the entire prospect.

In line with the reasoning that follows (5.5), we can consider the $AV@R$-agent as an insurant and the mean-variance agent as an insurer. From this point of view, what we obtain as optimal risk sharing under condition (5.8) is a *full-insurance* contract, where the insurer takes the whole risk X.

5.1.3 Sup-convolution of Non-Monotone Agents

We now compare the behaviour of two non-monotone agents in the problem of sharing a given total risk. We consider the most interesting cases, that is, the mean-variance and the standard-deviation criterions.

• `Mean-variance vs Standard-deviation`.

Proposition 5.6. *Let U_1 be the mean-variance principle (3.38) with parameter $\delta_1 > 0$, and let U_2 be the standard-deviation principle (3.39) with parameter $\delta_2 > 0$. Then, for any aggregate risk $X \in L^\infty$, there exists a unique (up to a constant) POA (X_1, X_2) given by*

$$(X_1, X_2) := (\alpha X, (1-\alpha)X), \tag{5.11}$$

where

$$\alpha = \begin{cases} \dfrac{\delta_2}{2\delta_1\sqrt{Var(X)}}, & \textit{if } \sqrt{Var(X)} \geq \dfrac{\delta_2}{2\delta_1}, \\ 1, & \textit{otherwise.} \end{cases} \tag{5.12}$$

This leads to the explicit calculation of the sup-convolution functional:

$$U(X) = U_1 \Box U_2(X) = \begin{cases} U_2(X) + \dfrac{\delta_2^2}{4\delta_1}, & \text{if } \sqrt{\mathrm{Var}(X)} \geq \dfrac{\delta_2}{2\delta_1}, \\ U_1(X), & \text{otherwise,} \end{cases} \tag{5.13}$$

where U satisfies Assumption 3.20, law-invariance and property (S), by Lemma 4.5.

Proof. As before, V_1 strictly convex implies $V = V_1 + V_2$ strictly convex on its effective domain. Therefore, for any fixed $X \in L^\infty$, there exists $Z_X \in \mathrm{dom}(V)$ such that $\partial U(X) = \{Z_X\}$, which in particular implies

$$X \in -\partial V(Z_X) = \Big\{ -\frac{Z_X}{2\delta} - cZ_X + d : \, c \in \mathbb{R}_0^+, \, d \in \mathbb{R} \Big\} = \{ -cZ_X + d : \, c \in \mathbb{R}^+, \, d \in \mathbb{R} \},$$

so that Z_X is, pointwise, a linear decreasing function of X. Therefore, for any POA $(X_1, X_2) \in \mathbb{A}_2(X)$,

$$X_1 = -\frac{Z_X}{2\delta} + d_1, \quad \text{for some } d_1 \in \mathbb{R},$$

and

$$X_2 = -c_2 Z_X + d_2, \quad \text{for some } c_2 \in \mathbb{R}_0^+ \text{ and } d_2 \in \mathbb{R},$$

thus showing the uniqueness (up to a constant) of optimal solutions in $\mathbb{A}_2(X)$. Moreover, we have that pointwise X_1 and X_2 are linear and non-decreasing functions of the total risk

X, which leads to (5.11) with $\alpha \in]0,1]$ (we exclude the trivial case where X is constant). At this point, the optimal parameter α is drawn out from the problem

$$\sup_{\alpha\in]0,1]} \{U_1(\alpha X) + U_2((1-\alpha)X)\} = \sup_{\alpha\in]0,1]} \{\mathbf{E}[X] - \delta_1\alpha^2\mathrm{Var}(X) - \delta_2(1-\alpha)\sqrt{\mathrm{Var}(X)}\},$$

which is equivalent to solve

$$\min_{\alpha\in]0,1]} \{\delta_1\alpha^2\sqrt{\mathrm{Var}(X)} + \delta_2(1-\alpha)\},$$

since X is not constant. This gives us exactly (5.12) and therefore

$$\begin{aligned} U(X) &= \begin{cases} U_1\Big(\dfrac{\delta_2 X}{2\delta_1\sqrt{\mathrm{Var}(X)}}\Big) + U_2\Big(\Big(1 - \dfrac{\delta_2}{2\delta_1\sqrt{\mathrm{Var}(X)}}\Big)X\Big), & \text{if } \sqrt{\mathrm{Var}(X)} \geq \dfrac{\delta_2}{2\delta_1}, \\ U_1(X), & \text{otherwise,} \end{cases} \\ &= \begin{cases} U_2(X) + \dfrac{\delta_2^2}{4\delta_1}, & \text{if } \sqrt{\mathrm{Var}(X)} \geq \dfrac{\delta_2}{2\delta_1}, \\ U_1(X), & \text{otherwise.} \end{cases} \end{aligned}$$

□

Note that (5.11) is a classical type of contract of coinsurance (called *quota-share contract*), where agents proportionally share the risk.

• `Mean-variance vs Mean-variance`.

The case of two (or more) agents endowed with mean-variance choice criterions (3.38) with parameters $\delta_i > 0$, is a particular case of dilated utility measures (4.7). Indeed, it is sufficient to take as risk-tolerance coefficients $\alpha_i = 1/\delta_i$ and as functional $\bar{U}$ the mean-variance principle with parameter $\delta = 1$, to have

$$U^{mv}_{\delta_i}(X) = \alpha_i U^{mv}_1\Big(\frac{X}{\alpha_i}\Big).$$

We know that, up to constants summing up to zero, there exists a unique solution $(X_i)_i$ to the sup-convolution problem and, by Proposition 4.15, it is given by

$$X_i = \frac{\alpha_i X}{\sum_j \alpha_j} = \frac{X}{\sum_j \delta_i/\delta_j}, \quad \text{for any } i.$$

This means that each agent i assumes a share of risk inversely proportional to her coefficient δ_i of aversion to the variability.

• `Standard-deviation vs Standard-deviation`.

We no longer have the same situation when considering two (or more) agents endowed with standard-deviation choice functionals (3.39) with different parameters $\delta_i > 0$. Here optimization leads to charging the total risk to the agent with the smallest parameter δ_i. To give a sketch of this, it is sufficient to note that, from relations $X \in -\partial V(Z) = -\sum_i \partial V_i(Z)$ and $X_i \in -\partial V_i(Z)$, we get

$$X = -aZ + b \quad \text{and} \quad X_i = -a_i Z + b_i,$$

for some $a, a_i \in \mathbb{R}_0^+$ and $b, b_i \in \mathbb{R}$ with $\sum_i a_i = a$ and $\sum_i b_i = b$. Moreover, $Z \in \cap_i \partial U_i(X_i)$ implies $\mathrm{Var}(Z) = \delta_i^2$ whenever X_i is not constant (i.e. $a_i \neq 0$), by Theorem 3.29. Now, since the parameters δ_i are different, for any possible choice of Z we have that the total risk X is taken by the agent i^* with parameter $\delta_{i^*}^2 = \mathrm{Var}(Z)$ (we exclude the trivial case where X is constant). Therefore, the overall utility is equal to $U(X) = \square_i U_i(X) = U_{i^*}(X)$. From this fact clearly follows that the unique (up to constants summing up to zero) POA is $(X_i)_i$ such that $X_i = 0$ for any i with $\delta_i > \min_j \delta_j$.

5.1.4 Semi-Deviation Agent vs Non-Monotone Agents

• `Semi-deviation vs Standard-deviation`.

Let us consider the semi-deviation utility with $p = 2$ and parameter $\delta \in (0, 1]$, and the standard-deviation principle with the same parameter δ. Once again we call them U_1 and U_2 respectively, and we note that $U_1(\xi) \geq U_2(\xi)$ for any $\xi \in L^\infty$, where the equality holds if and only if ξ is constant. From this fact and by the positive homogeneity of U_1, we obtain that the risk X is totally charged to the first agent. Indeed, for any POA $(X_1, X_2) \in \mathbb{A}_2(X)$, we get

$$\begin{aligned} U_1 \square U_2(X) &= U_1(X_1) + U_2(X_2) \leq U_1(X_1) + U_1(X_2) \\ &\leq U_1(X_1 + X_2) = U_1(X), \end{aligned}$$

the first inequality being strict whenever X_2 is non-constant. Therefore, up to a constant, the unique POA for any aggregate risk X, is $(X_1, X_2) := (X, 0)$.

• `Semi-deviation vs Mean-variance`.

Consider now the semi-deviation utility with $p = 2$ and parameter $\delta \in (0, 1]$, and the mean-variance principle with the same parameter δ. Once again there exists a unique (up to a constant) POA, characterized by one of the following situations: either the total risk is entirely taken by the mean-variance agent or, by means of (3.30) and Theorem 3.28, we have the interval $[\operatorname{ess\,inf} X, \operatorname{ess\,sup} X]$ shared in the two subintervals $[\operatorname{ess\,inf} X, \beta)$ (where the risk is proportionally shared between the agents) and $[\beta, \operatorname{ess\,sup} X]$ (where the risk is totally charged to the semi-deviation agent).

5.2 Optimal Risk Sharing: the Case of Three Agents

After the discussions of the previous section, it becomes easier to solve sup-convolution problems involving more than two agents. Here we present some cases involving three economic agents, calling to mind that the sup-convolution operator is associative.

• `AV@R vs Entropic vs Mean-variance`.

Proposition 5.7. *Let U_1 be the AV@R-criterion (3.14) with parameter $\lambda \in (0, 1]$, U_2 the entropic utility (3.17) with parameter $\gamma > 0$, and U_3 the mean-variance principle (3.38) with parameter $\delta > 0$. Then, for any aggregate risk $X \in L^\infty$, there exists a unique (up to constants summing up to zero) POA (X_1, X_2, X_3), such that $X_1 = -(X - k)^-$ and X_2 (resp. X_3) is a convex (resp. concave) function of $X \vee k$, for some $k \in \mathbb{R}$.*

Proof. Let us consider this problem in the following way:

$$U_1 \Box U_2 \Box U_3(X) = U_1 \Box (U_2 \Box U_3)(X) = U_1 \Box U(X),$$

where functional $U := U_2 \Box U_3$ results to be law-invariant, strictly monotone and strictly risk-averse conditionally on any event, by Lemma 4.5. Proposition 3.2 in [49] provides the unique (up to a constant) POA (ξ_1, ξ_2) of X with respect to (U_1, U), which is given by

$$(\xi_1, \xi_2) = (-(X - k)^-, X \vee k), \quad \text{for some } k \in \mathbb{R}.$$

This means that the interval of essential oscillations of X, $[\operatorname{ess\,inf} X, \operatorname{ess\,sup} X]$, can be shared in two subintervals $[\operatorname{ess\,inf} X, x]$, $[x, \operatorname{ess\,sup} X]$, such that ξ_2 is constant on the first one and ξ_1 on the second one. Therefore, the $AV@R$-agent takes the worst risks (the whole risk in $[\operatorname{ess\,inf} X, x]$), whereas the others agents share the rest of the risk. Now Proposition 5.1 gives us the recipe to optimally share the risk $\xi_2 = X \vee k$ between an entropic and a mean-variance agent. In conclusion, as POAs $(X_1, X_2, X_3) \in \mathbb{A}_3(X)$ w.r.t. (U_1, U_2, U_3), we obtain:

$$X_1 = -(X-k)^- + c_1,$$

$$X_2 = -\gamma \ln(g(\xi_2)) + c_2 = -\gamma \ln(g(X \vee k)) + c_2,$$

$$X_3 = -\frac{g(\xi_2)}{2\delta} + c_3 = -\frac{g(X \vee k)}{2\delta} + c_3,$$

for some $k \in \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ convex and decreasing function, and for any $c_1, c_2, c_3 \in \mathbb{R}$ s.t. $\sum_{i=1}^{3} X_i = X$. □

• `AV@R vs Mean-variance vs Semi-Deviation`.

Consider the $AV@R$-criterion with parameter $\lambda \in (0,1]$, the semi-deviation utility with $p = 2$ and parameter $\delta \in (0,1]$, and the mean-variance principle with the same parameter δ. As in the previous case, we can apply Proposition 3.2 in [49], since the sup-convolution of the last two agents provides a monetary utility functional, strictly monotone and strictly risk averse conditionally on lower tail events. In this way we obtain that the $AV@R$-agent assumes the worst risks, whereas the other agents share the rest of the risk as described in the previous section.

• `AV@R vs Standard-Deviation vs Semi-Deviation`.

Here we consider the same parameter δ for the standard-deviation principle and the semi-deviation utility (with $p = 2$), and proceed with the same reasoning as before, associating the last two agents in the first place. Once again the interval of the essential oscillations of the total risk is shared in two parts: the worst risks are entirely taken by the $AV@R$-agent, whereas the lowest risks are entirely taken by the semi-deviation agent.

• `AV@R vs Mean-variance vs Standard-Deviation`.

Proposition 5.8. *Let U_1 be the $AV@R$-criterion with parameter $\lambda \in (0,1]$, U_2 the mean-variance principle with parameter $\delta_1 > 0$, and U_3 the standard-deviation principle with* *parameter* $\delta_2 > 0$. *Then, for any aggregate risk* $X \in L^\infty$, *there exists a unique (up to constants summing up to zero) POA* (X_1, X_2, X_3), *given by*

$$(X_1, X_2, X_3) := (-(X-l)^- + (X-u)^+, \alpha((l \vee X) \wedge u), (1-\alpha)((l \vee X) \wedge u)), \quad (5.14)$$

for some $l, u \in \mathbb{R}$ *and*

$$\alpha = \begin{cases} \dfrac{\delta_2}{2\delta_1 \sqrt{Var((l \vee X) \wedge u)}}, & \text{if } \sqrt{Var((l \vee X) \wedge u)} \geq \dfrac{\delta_2}{2\delta_1}, \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Once again, we associate in the following way:

$$U_1 \Box U_2 \Box U_3(X) = U_1 \Box (U_2 \Box U_3)(X) = U_1 \Box U(X),$$

where we know the explicit form of the functional $U := U_2 \Box U_3$, described in (5.13), that satisfies all the requirements necessary to apply Proposition 5.3. Therefore, the unique (up to a constant) POA (ξ_1, ξ_2) for (U_1, U) is given by

$$(\xi_1, \xi_2) := (-(X-l)^- + (X-u)^+, (l \vee X) \wedge u), \text{ for some } l, u \in \mathbb{R}.$$

This means that the $AV@R$-agent takes the extremal risks, whereas $\xi_2 = (l \vee X) \wedge u$ is charged to the mean-variance and the standard-deviation agents. From Proposition 5.6 we know that these agents share risks proportionally between themselves, thus producing exactly (5.14).

□

Remark 5.9. *In all the preceding cases involving AV@R-agents, we have seen that they generally take the extreme risks, which reveals their non-conservative behaviour. This produces an opposite situation to that of the entropic-agents, which we have shown to be prudent towards extreme risks (compare Remark 5.2).*

• `Entropic vs Mean-variance vs Standard-Deviation`.

Proposition 5.10. *Let* U_1 *be the entropic utility with parameter* $\gamma > 0$, U_2 *the mean-variance principle with parameter* $\delta_1 > 0$, *and* U_3 *the standard-deviation principle with parameter* $\delta_2 > 0$. *Then, for any aggregate risk* $X \in L^\infty$, *there exists a unique (up to constants summing up to zero) POA* (X_1, X_2, X_3), *such that* X_1 *is a convex function of* X, *whereas* X_2 *and* X_3 *are concave functions of* X.

Proof. We proceed as in the preceding section, denoting U as the result of the convolution:

$$U(X) := U_1 \square U_2 \square U_3(X), \ \forall X \in L^\infty,$$

and V_1, V_2, V_3, V the convex conjugate functions of U_1, U_2, U_3, U respectively, with

$$V(\mu) = (V_1 + V_2 + V_3)(\mu), \ \forall \mu \in (L^\infty)^*,$$

strictly convex on its effective domain. By the characterization of the differentials found in the previous chapters, we obtain

$$X \in -\partial V(Z_X) = \{-\gamma \ln Z_X - cZ_X + d : \, c \in \mathbb{R}^+, \, d \in \mathbb{R}\},$$

where Z_X is the unique element in $\partial U(X)$. At this point, the POAs are given by

$$X_1 = -\gamma \ln(g(X)) + c_1,$$

$$X_2 = -\frac{g(X)}{2\delta_1} + c_2,$$

$$X_3 = -c_3 g(X) + c_4,$$

for some $c_3 \in \mathbb{R}_0^+$, g convex and decreasing function, and for all $c_1, c_2, c_4 \in \mathbb{R}$ such that $\sum_{i=1}^3 X_i = X$. □

CONCLUSIONS

For choice functionals satisfying Assumption 3.20, we obtain the very same results as Jouini et al. [49] for the existence and the characterization of the solutions to the optimal risk sharing problem (i.e. Pareto optimal allocations and optimal risk sharing rules). In particular, due to the cash-invariance property, we can solve it in two separate steps: first we maximize the overall utility which produces the Pareto optimal allocations (that is, how to re-share the aggregate risk: the shape of the contract); successively, we impose the individual rationality constraints, which denote the incentive for any agent to change her initial position and produce the indifference prices (that is, how much agents are willing to pay to enter into the transaction: the price of the contract). The introduction of the monotone approximation of non-monotone functionals, allows us to compare non-monotone agents with their monotone adjusted versions when facing this problem. In particular, provided that one agent is characterized by monotone preferences, we find a strict link between the solutions to the original ORS problem and the solutions to the new one only involving monotone choice functionals. We especially obtain interesting results when dealing with non-monotone agents having mean-variance preferences. In this situation, the Pareto optimal redistribution of the total risk is not sensitive to the lack of monotonicity by some agents, that is, to solve the sup-convolution problem w.r.t. the mean-variance criterions $U^{mv}_{\delta_i}$'s or w.r.t. the monotone-mean-variance criterions $U^{mmv}_{\delta_i}$'s turns out to be equivalent. Furthermore, we prove that the optimal redistribution of the risk often leads to simple contracts consisting in the exchange of European options written on the total risk or in a proportional sharing of it. In this way we get typical forms of insurance contracts, such as stop-loss and quota-share rules.

Part II

Absolutely Continuous Optimal Martingale Measures

Chapter 6

Expected Utility Maximization Problem

From now on, we consider the trading interval $[0, T]$, with $T \in (0, +\infty]$, as meaning that trading is possible at any date $t \in [0, T)$. The financial market consists in $(d+1)$ traded assets, whose prices at time t are described by the random variables $\tilde{S}_t^0, \tilde{S}_t^1, \ldots, \tilde{S}_t^d$, measurable with respect to $\mathcal{F}_t$. The asset indexed by 0 is the riskless one (that we consider strictly positive), and without loss of generality we make the choice to deal with the vector of discounted prices $S_t = (S_t^1, \ldots, S_t^d)$, where $S_t^i = \tilde{S}_t^i / \tilde{S}_t^0$ (that is, we choose the cash account as numéraire). In particular, we assume the $\mathbb{R}^d$-valued stochastic process $S = (S_t)_{0 \le t \le T}$ to be a locally-bounded semimartingale based on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$ (see §2).

Here a trading strategy is defined as a stochastic process $H = ((H_t^1, \ldots, H_t^d))_{0 \le t \le T}$ in $\mathbb{R}^d$, where H_t^i denotes the number of shares of asset i held in the portfolio at time t. In order to rule out doubling strategies, we consider the following set of admissible trading rules (introduced in [42], see also [24]):

$$\mathcal{H} := \{H : H \text{ predictable and } S\text{-integrable}, \; H \cdot S \text{ uniformly bounded from below}\},$$

where $H \cdot S$ denotes the stochastic integral of H with respect to S: $(H \cdot S)_t := \displaystyle\int_0^t H_u dS_u$ (for details on stochastic integration we refer to [47], [66], [68]).

6.1 The Primal and the Dual Problem

At this point, for an agent with initial endowment $x \in \mathbb{R}$ and preferences described by a utility function u, the expected utility maximization problem in the market S can be written as follows:

$$w(x) := \sup_{H \in \mathcal{H}} \mathbf{E}[u(x + (H \cdot S)_T)]. \tag{6.1}$$

We work in an incomplete market and allow the wealth processes to be negative, so that the utility functions we consider are defined and finitely valued on the entire real line. In this setting, basic results of existence and uniqueness of the optimal solution are given in Schachermayer [69], to which we refer for a complete outline of the situation. In order to apply these results, we need to make some assumptions on the market S and on the utility function u. In particular, we assume the utility function to behave according to the following technical requirements:

Assumption 6.1. *$u : \mathbb{R} \to \mathbb{R}$ is a smooth, strictly increasing and strictly concave function satisfying the following conditions:*

(I) Inada conditions: $\lim_{x \to -\infty} u'(x) = \infty$ *and* $\lim_{x \to +\infty} u'(x) = 0$*;*

(II) Reasonable Asymptotic Elasticity (RAE) conditions:

$$\liminf_{x \to -\infty} \frac{xu'(x)}{u(x)} > 1 \qquad \textit{and} \qquad \limsup_{x \to +\infty} \frac{xu'(x)}{u(x)} < 1.$$

Together with the classical assumption (I) on the marginal utility, we require (II) on the ratio between the marginal and the average utility (suggested by economic intuition, see [69] and [71] for the precise meaning and for a connection with the relative risk aversion) since it is the crucial condition to ensure the existence of the optimal investment. We recall that a typical example of function satisfying these conditions is the exponential utility $u(x) = -\exp(-x)$.

Now we need to introduce two particular sets of probability measures. Therefore, we denote by $\mathcal{M}^a(S)$ the family of absolutely continuous local martingale measures:

$$\mathcal{M}^a(S) := \{\mathbb{Q} \ll \mathbb{P} : \mathbb{Q} \text{ is a probability measure and } S \text{ is a local martingale under } \mathbb{Q}\},$$

and by $\mathcal{M}^e(S)$ the family of equivalent local martingale measures (also called risk-neutral measures):

$$\mathcal{M}^e(S) := \{\mathbb{Q} \sim \mathbb{P} : \mathbb{Q} \text{ is a probability measure and } S \text{ is a local martingale under } \mathbb{Q}\}.$$

This allows us to formulate the classical assumption of "no riskless-profits available in the market" in the following suitable version:

Assumption 6.2. *The set $\mathcal{M}^e(S)$ is not empty.*

This condition insures that in the market modelled by S there are *no* possibilities of *free lunch with vanishing risk* (NFLVR), which is a slight generalization of the *no-arbitrage* (NA) condition and can be expressed as follows: there exists no sequence of terminal payoffs of admissible integrands, $f_n = (H^n \cdot S)_T$, such that the negative parts f_n^- tend to 0 uniformly, and such that f_n tends almost surely to a non-negative function f_0 satisfying $\mathbb{P}(f_0 > 0) > 0$ (see [24] for this version of the Fundamental Theorem of Asset Pricing).

Under this assumption, we know that a complete market S is characterized by the existence of a unique element in $\mathcal{M}^e(S)$. Every contingent claim is attainable here, that is, there exists an admissible trading strategy which perfectly replicates it, and therefore we have a simple pricing and hedging theory. In particular, since the price of a contingent claim is uniquely determined by no-arbitrage arguments, there is no need to involve agents' preferences. On the contrary, in incomplete markets we have many equivalent martingale measures, which correspond to many linear pricing rules all compatible with the (NA) condition, so that the problem that arises is how to select a measure among them (compare [34] and the references therein). For example, one may choose to consider the so-called minimax martingale measure (see [6], [34]), even though, in our framework, we obtain no answers in this direction (compare Remark 7.3).

To exclude the trivial degenerate case, we make a further, intuitive requirement involving both u and S:

Assumption 6.3. *For any stopping time $\rho \in [0, T]$,*

$$\sup_{H = H1_{]\rho, T]}} \boldsymbol{E}[u((H \cdot S)_T) | \mathcal{F}_\rho] < u(\infty) \ a.s.\,. \tag{6.2}$$

In particular this condition serves to insure the finiteness of the value function w in (6.1), on all the real line.

As said in the Introduction, many authors solve maximization problems by using duality methods. Here we make the same choice, and this leads to the characterization of the maximizer in (6.1) in terms of the optimal solution to a dual variational problem. In order to do this, we need to introduce the convex conjugate $v : \mathbb{R}^+ \to \mathbb{R}$ of the utility function u:

$$v(y) = \sup_{x\in\mathbb{R}}(u(x) - xy), \quad \forall y > 0. \tag{6.3}$$

Remark 6.4. *Note that, since we are dealing with an increasing function u on $\mathbb{R}$, the duality automatically works in $\mathbb{R}^+$. Indeed, it is obvious that if we consider a negative y in (6.3) we obtain $v(y) = +\infty$.*

Under our assumptions, v turns out to be a smooth and strictly convex function such that

$$v(0) = u(+\infty),\ v(+\infty) = +\infty \quad \text{and} \quad v'(0) = -\infty,\ v'(+\infty) = +\infty.$$

At this point, the optimal problem dual to (6.1) can be expressed in the following way:

$$\nu(y) := \inf_{\mathbb{Q}\in\mathcal{M}^a(S)} \mathbf{E}\Big[v\Big(y\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big], \tag{6.4}$$

where the function $\nu : \mathbb{R}^+ \to \mathbb{R}$ is finite from Assumption 6.3. For example, for $u(x) = -\exp(-x)$ we obtain $v(y) = y(\ln y - 1)$ and $\nu(y) = v(y) + y \inf_{\mathbb{Q}\in\mathcal{M}^a(S)} H(\mathbb{Q};\mathbb{P})$. In this case, the dual problem (6.4) provides the local martingale measure with the minimal relative entropy (3.18) (compare [34]). We now define the *generalized entropy* with respect to $\mathbb{P}$ as the function acting as follows:

$$\mathbb{Q} \longmapsto \mathbf{E}\Big[v\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big],$$

for any measure $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$. By means of this function we can introduce two particular sets of local martingale measures for the market S:

$$\mathcal{M}^a_f(S) := \Big\{\mathbb{Q} \in \mathcal{M}^a(S) : \mathbf{E}\Big[v\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big] < \infty\Big\}$$

and

$$\mathcal{M}^e_f(S) := \{\mathbb{Q} \in \mathcal{M}^e(S) : \mathbf{E}\Big[v\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big] < \infty\},$$

which allow us to formulate the dual problem (6.4) on $\mathcal{M}^a_f(S)$ instead of $\mathcal{M}^a(S)$.

In our setting we can apply a result of Bellini and Frittelli [6] which ensures that the unique minimizer in (6.4) (the so-called *minimax martingale measure*) exists and belongs to $\mathcal{M}^a_f(S)$. Define $\widehat{X}_T(x)$ as the optimal terminal wealth, unique solution to the primal problem (6.1), and $\widehat{\mathbb{Q}}_y$ as the minimal martingale measure, unique solution to the dual problem (6.4). The basic idea of the dual approach is to solve the latter problem and then, by convex duality, to solve the former one. The crucial formula which relates the respective optimizers is given by

$$\frac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}} = \frac{u'(\widehat{X}_T(x))}{y}, \tag{6.5}$$

where $y = w'(x) > 0$ (see Theorem 7.1 below).

6.2 Equivalent Case and Absolutely-Continuous Case

Let us now outline the direction our study moves in. By solving the dual problem, two mutually exclusive situations are singled out:

- *equivalent case*: $\widehat{\mathbb{Q}} \in \mathcal{M}^e(S)$;
- *absolutely-continuous case*: $\widehat{\mathbb{Q}} \in \mathcal{M}^a(S) \setminus \mathcal{M}^e(S)$.

Here and in what follows, where it does not generate confusion, we do not indicate the dependence -of the optimal solutions- on the initial capital x that the agent is endowed with.

Remark 6.5. *Note that the absolutely-continuous case corresponds to have an optimal terminal wealth which is infinite with strictly positive probability. Indeed, let $A \in \mathcal{F}$ denote the maximal set such that $\widehat{\mathbb{Q}}(A) = 0$. By relation (6.5) and the Inada conditions, we have $A = \left\{\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} = 0\right\} = \{\widehat{X}_T = +\infty\}$ $\mathbb{P}$-almost surely. On the other hand, the absolutely-continuous case clearly implies $\mathbb{P}(A) > 0$, thus giving $\mathbb{P}(\{\widehat{X}_T = +\infty\}) > 0$.*

As said in the Introduction, authors often assume the minimax martingale measure to belong to $\mathcal{M}^e(S)$, so that the results are given in the equivalent case. This is what happens, for example, when in [69] the solution to the primal problem is shown to be equal to the final wealth of some self-financing strategy (Theorem 7.1 below). More precisely, supposing to be in the equivalent case, we obtain the optimal wealth at the time horizon T as the terminal value of a $\widehat{\mathbb{Q}}_{w'(x)}$-uniformly integrable martingale: $\widehat{X}_T(x) = x+(\widehat{H}(x)\cdot S)_T$. On the contrary, if the minimax martingale measure $\widehat{\mathbb{Q}}$ is just absolutely continuous with respect to $\mathbb{P}$, we lose this characterization, in the sense that the integral representation of $\widehat{X}_T$ holds true under the optimal measure only (i.e., $\widehat{\mathbb{Q}}$-almost surely). In our study we focus on the absolutely-continuous case and show how to reach the wealth optimal at time T by means of new problems, which are defined in some random intervals contained in $[0, T]$ and fit in with the equivalent case. Before doing so, we mention some known situations in which this fact cannot occur, that is, some conditions which separately ensure that $\widehat{\mathbb{Q}} \sim \mathbb{P}$:

1. $u(+\infty) = +\infty$;
2. $\mathcal{M}^e_f(S) \neq \emptyset$;
3. finite-discrete-time market model.

In the first case we have $v(0) = u(+\infty) = +\infty$, and the formulation (6.4) of the dual problem makes the minimizer $\widehat{\mathbb{Q}}$ satisfying $\dfrac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} > 0$ $\mathbb{P}$-almost surely, i.e., the optimal measure lies in the set $\mathcal{M}^e(S)$.

On the other hand, if there exists an equivalent martingale measure with finite generalized entropy, since we have assumed the Inada conditions to hold, once again we obtain the measure $\widehat{\mathbb{Q}}$ to be equivalent to $\mathbb{P}$ (see Csiszar [16] for the exponential utility and Kabanov-Stricker [50] for the general case).

Lastly, the case of a market model with finite discrete time always falls into one of the two previous situations, thus fitting the equivalent case. Indeed, suppose $v(0) = u(+\infty) < +\infty$. In this case the Dalang-Morton-Willinger theorem applies (see [17]), since we work under the (NA) condition, yielding an equivalent martingale measure with bounded density. In particular, this implies that this martingale measure lies in $\mathcal{M}^e_f(S)$ and concludes what was previously declared.

Chapter 7

Approximation of the Optimal Wealth

In this chapter we study a characterization of the solution to the optimization problem (6.1) by means of new problems obtained suitably stopping the original one. These auxiliary problems allow us to give convergence results in Section 7.3. In particular, the optimal wealth $\widehat{X}_T$ turns out to be achievable as the limit of terminal values for some self-financing trading strategies.

7.1 The Original Problem

First it is convenient to state some known results we rely heavily on. In view of this, we define the strictly decreasing function $I : \mathbb{R}^+ \to \mathbb{R}$ given by

$$I(y) = (u')^{-1}(y) = -v'(y),$$

which produces $v(y) = u(I(y)) - yI(y)$.

Theorem 7.1. *[69, Theorem 2.2] Let the locally-bounded semimartingale $S = (S_t)_{0\leq t\leq T}$ and the utility function $u : \mathbb{R} \to \mathbb{R}$ satisfy Assumptions 6.1-6.3. Then*

(i) The value functions w and ν defined in (6.1) and (6.4) are conjugate; they are finitely valued, continuously differentiable, strictly concave (resp. convex) on $\mathbb{R}$ (resp. $\mathbb{R}^+$)

and satisfy

$$w'(-\infty) = -\nu'(0) = \nu'(+\infty) = +\infty, \quad w'(+\infty) = 0. \tag{7.1}$$

(ii) The optimizers $\widehat{X}_T(x)$ and $\widehat{\mathbb{Q}}_y$ in (6.1) and (6.4) exist, are unique and satisfy

$$\widehat{X}_T(x) = I\Big(y\frac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}\Big) \text{ or, equivalently, } \frac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}} = \frac{u'(\widehat{X}_T(x))}{y}, \tag{7.2}$$

where $x \in \mathbb{R}$ and $y \in \mathbb{R}^+$ are related via $y = w'(x)$.

(iii) The following relations hold true:

$$xw'(x) = \boldsymbol{E}[\widehat{X}_T(x)u'(\widehat{X}_T(x))], \quad \nu'(y) = \boldsymbol{E}\Big[\frac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}v'\Big(y\frac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}\Big)\Big]. \tag{7.3}$$

(iv) If $\widehat{\mathbb{Q}}_y \in \mathcal{M}^e(S)$ and $x = -\nu'(y)$, then $\widehat{X}_T(x)$ equals the terminal value of a process of the form $\widehat{X}_t(x) = x + (\widehat{H}(x) \cdot S)_t$, where $\widehat{H}$ is a predictable and S-integrable trading strategy such that $\widehat{X}(x)$ is a uniformly integrable martingale under $\widehat{\mathbb{Q}}_y$.

From these formulae we also obtain the following one:

$$w(x) = \mathbf{E}[u(\widehat{X}_T(x))] = xy + \mathbf{E}\Big[v\Big(y\frac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}\Big)\Big], \ \ y = w'(x), \tag{7.4}$$

that we use to formulate the primal problem (6.1) in a different way. In this order, we also introduce a suitable version of a proposition proved by Biagini and Frittelli [7] in a more general context:

Proposition 7.2. *Let u satisfy Assumption 6.1 and $\mathbb{Q}$ be any measure in $\mathcal{M}^a_f(S)$. Then, for $x \in \mathbb{R}$, the optimal solution to*

$$\min_{\lambda>0} \lambda x + \boldsymbol{E}\Big[v\Big(\lambda\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big] \tag{7.5}$$

is the unique solution to the first order condition

$$x + \boldsymbol{E}\Big[\frac{d\mathbb{Q}}{d\mathbb{P}}v'\Big(\lambda\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big] = 0. \tag{7.6}$$

Noting that relation (7.4) makes $y = w'(x)$ solving (7.6) for $\mathbb{Q} = \widehat{\mathbb{Q}}_y$, we get $w'(x)$ as the optimizer to the problem

$$\min_{\lambda>0} \lambda x + \mathbf{E}\Big[v\Big(\lambda\frac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}\Big)\Big]. \tag{7.7}$$

This allows us to rewrite the maximization problem (6.1) as

$$w(x) = \min_{\lambda>0, \mathbb{Q}\in\mathcal{M}_f^a(S)} \lambda x + \mathbf{E}\Big[v\Big(\lambda\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big], \tag{7.8}$$

which admits unique solution $(\lambda, \mathbb{Q}) = (y, \widehat{\mathbb{Q}}_y)$, where as usual $y = w'(x)$. In what follows it will turn out to be convenient to consider this formulation of the primal problem (6.1), which also involves the dual optimizer.

Let us now take a look at the integral representation in Theorem 7.1-*(iv)*. What we get is a perfect replicability of the optimal terminal wealth, by trading in the market in accordance with a self-financing strategy $\widehat{H}$. As previously mentioned, this characterization of the optimal wealth is submitted to the equivalence of $\widehat{\mathbb{Q}}$ to $\mathbb{P}$, whereas the non-equivalent (i.e., the absolutely-continuous) case is left open. Our main goal here is to provide an approximation of the optimal solution to the problem (6.1) by means of solutions to auxiliary maximization problems, solutions which admit integral representation. Therefore we often assume, or emphasize, the case where the minimax martingale measure is not equivalent to the historical probability, which is the situation where our convergence results become meaningful.

Remark 7.3. *In the absolutely-continuous case, we lose not only the integral representability of the optimal terminal wealth, but also the use of this dual approach as a methodology to give an answer to the problem of selecting a (NA)-compatible pricing measure. Indeed, in this case, the linear pricing rule given by* $\boldsymbol{E}_{\widehat{\mathbb{Q}}}[.]$ *is not positive and leads to arbitrage opportunities.*

We now impose a further requirement on our market model, bearing in mind that this condition is satisfied, for example, by the brownian filtration.

Assumption 7.4. *Every stopping time is* $(\mathcal{F}_t)$*-predictable.*

7.2 The Auxiliary Problems

As pointed out, we introduce a sequence of optimization problems which lead to the approximation of the optimal wealth $\widehat{X}_T$ in terms of final values of stochastic integrals. Obviously this becomes significant in the absolutely-continuous case, where $\widehat{X}_T$ is infinite with strictly positive probability and does not admit integral representation.

Denoting by $(\widehat{Z}_t)_{0\le t\le T}$ the density process corresponding to the optimal martingale measure $\widehat{\mathbb{Q}}$:

$$\widehat{Z}_0 \equiv 1, \quad \widehat{Z}_t = \mathbf{E}\Big[\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\Big|\mathcal{F}_t\Big], \forall t \in (0,T), \quad \widehat{Z}_T = \widehat{Z} = \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}, \tag{7.9}$$

we can define the following stopping times:

$$\tau = \inf\{t > 0 : \widehat{Z}_t = 0\}, \quad \tau_n = \inf\{t > 0 : \widehat{Z}_t \le n^{-1}\}, \forall n \in \mathbb{N}, \tag{7.10}$$

where we put $\inf \emptyset = +\infty$ and $\widehat{Z}_\infty = \widehat{Z}_\tau = \widehat{Z}_T$.

7.2.1 Announcing Stopping Times

By Assumption 7.4 τ is a predictable time, hence there exists a sequence $(\sigma_n)_{n\ge 1}$ of stopping times announcing it:

$$\sigma_n \text{ increasing}, \quad \sigma_n < \tau, \forall n \in \mathbb{N}, \quad \text{and} \quad \lim_{n\to\infty} \sigma_n = \tau.$$

We can show that τ is announced exactly by $(\tau_n)_{n\ge 1}$ on $\{\tau < \infty\}$.

Proposition 7.5. *Let us define the stopping times*

$$\overline{\tau}_n = \tau_n 1_{\{\tau_n<\infty\}} + n 1_{\{\tau_n=\infty\}}. \tag{7.11}$$

Under Assumption 7.4, the sequence $(\overline{\tau}_n)_n$ announces τ.

Proof. By (7.10) and (7.11) we have $\overline{\tau}_n$ increasing and $\overline{\tau}_n \le \tau_n \le \tau$. Moreover $\overline{\tau}_n < \tau$ clearly holds on $\{\tau = \infty\}$ (which equals Ω $\mathbb{P}$-a.s. if we are in the equivalent case). We now consider the absolutely-continuous case and prove that the strict inequality is also true in the set $\{\tau < \infty\}$. If not, there exists $B \subset \{\tau < \infty\}$ with $\mathbb{P}(B) > 0$ and $\tau_n = \overline{\tau}_n = \tau$ on B. Since $(\widehat{Z}_t)_t$ is a uniformly integrable martingale, the optional-sampling theorem and the martingale convergence theorem give us

$$\widehat{Z}_{\tau-} = \lim_n \widehat{Z}_{\sigma_n} = \lim_n \mathbf{E}[\widehat{Z}_\tau|\mathcal{F}_{\sigma_n}] = \mathbf{E}[\widehat{Z}_\tau|\vee_n\mathcal{F}_{\sigma_n}] = \mathbf{E}[\widehat{Z}_\tau|\mathcal{F}_{\tau-}] = 0$$

on $\{\tau < \infty\}$, τ being $\mathcal{F}_{\tau-}$-measurable. On the other hand, since $\overline{\tau}_n > 0$, we have $\widehat{Z}_{\overline{\tau}_n-} \ge n^{-1}$ by definition. We then obtain

$$0 = \widehat{Z}_{\tau-}1_B = \widehat{Z}_{\overline{\tau}_n-}1_B \ge n^{-1}1_B$$

and this contradiction proves $\overline{\tau}_n < \tau$ a.s. on Ω. In order to end the proof, there still remains to show that $\overline{\tau}_n$ (or, equivalently, τ_n) converges to τ. By monotonicity, $\eta = \lim_n \tau_n \leq \tau$ is well defined and, of course, $\eta \geq \tau_n$. Let us show that this limit just equals τ. Since on $\{\eta = \infty\}$ this is clearly true, we consider the set $\{\eta < \infty\}$ (where, $\forall n \in \mathbb{N}$, $\tau_n < \infty$ too). The optional-sampling theorem gives us

$$\mathbf{E}[\widehat{Z}_\eta 1_{\{\tau_n<\infty\}}] = \mathbf{E}[\widehat{Z}_{\tau_n} 1_{\{\tau_n<\infty\}}] \leq n^{-1},$$

since on $\{\tau_n = \infty\}$ we have $\eta = \tau_n = \infty$ and $\widehat{Z}_\eta = \widehat{Z}_{\tau_n} = \widehat{Z}_T$. Therefore, by applying Chebyshev's inequality, $\mathbb{P}(\{\widehat{Z}_\eta 1_{\{\tau_n<\infty\}} \geq c\}) \leq (cn)^{-1}$ for any constant $c > 0$ we fix. This yields

$$\lim_{n\to\infty} \mathbb{P}(\{\widehat{Z}_\eta 1_{\{\tau_n<\infty\}} \geq c\}) = 0, \quad \forall c > 0,$$

that is, $\widehat{Z}_\eta 1_{\{\tau_n<\infty\}}$ tends to 0 in probability. It follows from the dominated convergence that

$$\| \widehat{Z}_\eta 1_{\{\tau_n<\infty\}} \|_{L^1(\mathbb{P})} \to 0 \qquad \text{as} \quad n \to \infty,$$

hence we get $\widehat{Z}_\eta 1_{\{\eta<\infty\}} = 0$, since $\mathbf{E}[\widehat{Z}_\eta 1_{\{\eta<\infty\}}] \leq \mathbf{E}[\widehat{Z}_\eta 1_{\{\tau_n<\infty\}}]$. This fact leads us to conclude that $\eta = \tau$ by (7.10), and makes our proof complete.

□

This proposition clearly states the continuity of the density process $(\widehat{Z}_t)_t$ at τ. Indeed, the right continuity of the filtration yields the right continuity of any uniformly integrable martingale process, and the assertion of the proposition gives us

$$\lim_{t\uparrow\tau} \widehat{Z}_t = \lim_{n\to\infty} \widehat{Z}_{\tau_n} = \widehat{Z}_\tau.$$

We point out that, in the proof of Proposition 7.5, we actually do not use the specific fact that $(\widehat{Z}_t)_t$ is the density process of the dual minimizer, and this allows us to reformulate the proposition in a more general way. By doing so, we obtain the following result, interesting by itself.

Proposition 7.6. *Let $M = (M_t)_{0\leq t\leq T}$ be a non-negative uniformly integrable martingale in a filtered probability space $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$, with $T \in (0, \infty]$ and $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfying* *the usual conditions. Let ρ be the first time in which M reaches zero and assume that it is a predictable stopping time. Then ρ is announced by the sequence of stopping times*

$$\overline{\rho}_n = \rho_n 1_{\{\rho_n < \infty\}} + n 1_{\{\rho_n = \infty\}},\, n \in \mathbb{N}, \tag{7.12}$$

where

$$\rho_n = \inf\{t > 0 : M_t \leq n^{-1}\}. \tag{7.13}$$

Remark 7.7. *In addition to what we have concluded above, it is not unworthy to underline that $\widehat{X}_T(x), \widehat{Z}_T \in L^0(\Omega, \mathcal{F}_{\tau-}, \mathbb{P})$ and relation*

$$w_{\tau-}(x) := \sup_H \boldsymbol{E}[u(x + (H \cdot S)_{\tau-})] = w(x) \tag{7.14}$$

holds true. This means that we can visualize the optimization problem (6.1) in $[0,T]$, as it was defined in the random interval $[0,\tau[$. Of course, the optimizers of the relative dual problems coincide as well:

$$\nu_{\tau-}(y) := \inf_{\mathcal{M}^a(S^{\tau-})} \boldsymbol{E}\Big[v\Big(y\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big] = \nu(y),$$

where $y = w'(x) = w'_{\tau-}(x)$ and $\mathcal{M}^a(S^{\tau-})$ refers to the stopped process $(S_t^{\tau-})_{0\leq t\leq T} = (S_{t\wedge\tau-})_{0\leq t\leq T}$. Therefore, from now on, we regard problems (6.1) and (7.14) as indistinguishable.

7.2.2 Primal and Dual Problems

Let us consider the trading random interval $[0, \tau_n]$, for any n in $\mathbb{N}$. We define the expected utility maximization problem relative to it:

$$w_n(x) := \sup_{H\in\mathcal{H}} \mathbf{E}[u(x + (H \cdot S)_{\tau_n})], \quad x \in \mathbb{R}, \tag{7.15}$$

as well as the associated dual one:

$$\nu_n(y) := \inf_{\mathbb{Q}\in\mathcal{M}^a(S^{\tau_n})} \mathbf{E}\Big[v\Big(y\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big], \quad y \in \mathbb{R}^+, \tag{7.16}$$

where $\mathcal{M}^a(S^{\tau_n})$ refers to the stopped process $(S_t^{\tau_n})_{0\leq t\leq T} = (S_{t\wedge\tau_n})_{0\leq t\leq T}$. Of course $(w_n)_n$ is increasing and $w_n \leq w$. Note that (7.15) can be reformulated in the following equivalent ways:

$$w_n(x) = \sup_{H\in\mathcal{H}} \mathbf{E}[u(x + (H \cdot S^{\tau_n})_T)] = \sup_{H\in\mathcal{H}} \mathbf{E}[u(x + (H1_{[0,\tau_n]} \cdot S)_T)].$$

From now on we will use the notation $X_T^{(n)}(x) = X_{\tau_n}^{(n)}(x)$ and $\mathbb{Q}_y^{(n)}$ respectively, for the terminal wealth solving the primal problem (7.15) and for the martingale measure solving the dual problem (7.16) (the dependence on x or y being dropped when it does not generate confusion). It visibly follows that the optimal solutions relative to the problems in $[0,\tau_n]$, satisfy relations analogous to those in (7.2):

$$X_T^{(n)}(x) = I\Big(y_n \frac{d\mathbb{Q}_{y_n}^{(n)}}{d\mathbb{P}}\Big) \quad \text{and} \quad \frac{d\mathbb{Q}_{y_n}^{(n)}}{d\mathbb{P}} = \frac{u'(X_T^{(n)}(x))}{y_n}, \tag{7.17}$$

where $y_n = w_n'(x) = \mathbf{E}[u'(X_T^{(n)}(x))]$.

The following lemma is a simple observation and a key point in view of our convergence results.

Lemma 7.8. *The optimal martingale measure* $\mathbb{Q}_{y_n}^{(n)}$ *is equivalent to* $\mathbb{P}$ *for any* n *in* $\mathbb{N}$.

Proof. As mentioned in Section 6.2, under the Inada conditions, if there exists an equivalent martingale measure with finite generalized entropy, then the optimal measure is equivalent to the historical probability. We can show that, in fact, this is the case for the optimization problem in $[0,\tau_n]$. To this end it is sufficient to prove that the measure $\widehat{\mathbb{Q}}_y$ (optimal solution to the dual problem in $[0,T]$) restricted to the σ-algebra $\mathcal{F}_{\tau_n}$:

$$\frac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}\Big|_{\mathcal{F}_{\tau_n}} = \mathbf{E}\Big[\frac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}\Big|\mathcal{F}_{\tau_n}\Big],$$

belongs to $\mathcal{M}_f^e(S^{\tau_n})$. We clearly have $\frac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}\Big|_{\mathcal{F}_{\tau_n}} > 0$ a.s. by definition of τ and Proposition 7.5. Moreover, by Jensen's inequality we have

$$\mathbf{E}\Big[v\Big(y\frac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}\Big|_{\mathcal{F}_{\tau_n}}\Big)\Big] = \mathbf{E}\Big[v\Big(\mathbf{E}\Big[y\frac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}\Big|\mathcal{F}_{\tau_n}\Big]\Big)\Big] \leq \mathbf{E}\Big[v\Big(y\frac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}\Big)\Big] < \infty,$$

as claimed.

□

Therefore, the unique solution to the optimal problem (7.16) can be written in the following way

$$X_{\tau_n}^{(n)}(x) = x + (H^n \cdot S)_{\tau_n}, \tag{7.18}$$

by Theorem 7.1. Here $X^{(n)}_{\tau_n}(x)$ is finite $\mathbb{P}$-a.s., $H^n = H^n 1_{[0,\tau_n]}$ is a predictable S^{τ_n}-integrable process, and $(H^n \cdot S)_{0\le t\le \tau_n}$ is a $\mathbb{Q}^{(n)}_{y_n}$-uniformly integrable martingale. It is exactly this characterization of $X^{(n)}_{\tau_n}(x)$ that makes the convergence results proved in the next section interesting, as well argued after the statement of Theorem 7.9.

7.3 Convergence Results

Our main theorem in this setting can be stated as follows.

Theorem 7.9. *Assume $u : \mathbb{R} \to \mathbb{R}$ and $(S_t)_{0\le t\le T}$ to satisfy Assumptions 6.1-6.3, 7.4. Then the following relations between the solutions to the original problems and the auxiliary ones hold true:*

(i) $w_n(x) \underset{n}{\longrightarrow} w(x)$;

(ii) $X^{(n)}_T(x) \xrightarrow[n]{\mathbb{P}} \widehat{X}_T(x)$ *and* $\dfrac{d\mathbb{Q}^{(n)}_{y_n}}{d\mathbb{P}} \xrightarrow[n]{L^1(\mathbb{P})} \dfrac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}$.

This theorem states that the optimal wealth $\widehat{X}_T$ (solution to the original problem formulated in $[0,T]$) can be approximated through the optimal wealths $X^{(n)}_T$ (reachable by trading up to the random times τ_n only). Clearly the interesting case is the absolutely-continuous one, where $\widehat{X}_T$ admits integral representation only $\widehat{\mathbb{Q}}$-almost surely. At this point it is of fundamental importance to consider Theorem 7.9 in conjunction with Lemma 7.8, that is, with characterization (7.18) of the optimal wealths $X^{(n)}_{\tau_n}$, which makes $\widehat{X}_T$ attainable as the limit of suitable-portfolios terminal values.

Before proving Theorem 7.9, we need some preparatory results. Let us introduce the sequence of positive measures

$$(y_n Z^{(n)})_{n\in\mathbb{N}},$$

where $Z^{(n)} := \dfrac{d\mathbb{Q}^{(n)}_{y_n}}{d\mathbb{P}}$. We want to prove that we can extract a sequence of convex combinations of them which converges to $y\widehat{Z} = w'(x)\dfrac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}$ in probability, where as usual y and y_n are the first derivatives of the value functions w and w_n at a fixed point $x \in \mathbb{R}$. This is just a preliminary result, and it is only in the next subsection that we will be able to show that the sequence $(y_n Z^{(n)})_n$ itself converges to $y\widehat{Z}$.

Recall that functions u, w, w_n are increasing, concave, and finite valued on $\mathbb{R}$, and that sequence $(w_n)_n$ is increasing too. Moreover, relation $u \leq w_n \leq w$, $\forall n \in \mathbb{N}$, is clearly satisfied. Hence, for any fixed $x \in \mathbb{R}$, $(y_n)_n = (w'_n(x))_n$ is a bounded sequence, say $y_n \leq \xi \in \mathbb{R}\ \ \forall n \in \mathbb{N}$, where of course $\xi = \xi(x)$. It follows that $(y_n Z^{(n)})_n$ is a bounded sequence as well, lying in $L^1_+(\Omega, \mathcal{F}, \mathbb{P})$, and we can make use of an appropriate version of Komlos' theorem (see [55] and [25]). This produces a sequence $(g_n)_{n\in\mathbb{N}}$ of positive measures converging in probability to some $g \in L^1_+(\Omega, \mathcal{F}, \mathbb{P})$. More precisely we have

$$g_n = \sum_{k=n}^{\infty} \alpha_k^n y_k Z^{(k)} \in \text{ conv}\,(y_n Z^{(n)}, y_{n+1} Z^{(n+1)}, \ldots),\ \ n \in \mathbb{N}, \tag{7.19}$$

with $0 \leq \alpha_k^n \leq 1,\ \sum_{k=n}^{\infty} \alpha_k^n = 1,$ and $g_n \xrightarrow{\mathbb{P}} g \in L^1_+(\Omega, \mathcal{F}, \mathbb{P})$.

It is convenient to introduce the probability measures related to these random variables:

$$\frac{d\mathbb{R}^n}{d\mathbb{P}} = \frac{g_n}{\mathbf{E}[g_n]} = \frac{g_n}{\gamma_n}, \quad \frac{d\mathbb{R}}{d\mathbb{P}} = \frac{g}{\mathbf{E}[g]} = \frac{g}{\gamma}, \quad \gamma_n,\, \gamma \in (0, \infty). \tag{7.20}$$

As an immediate consequence of the boundedness of $(y_n)_n$, we have that $(\gamma_n)_n$ is bounded too. Indeed Fatou's lemma gives us

$$\gamma_n = \mathbf{E}\Big[\sum_{k=n}^{\infty} \alpha_k^n y_k Z^{(k)}\Big] \leq \sum_{k=n}^{\infty} \mathbf{E}[\alpha_k^n y_k Z^{(k)}] = \sum_{k=n}^{\infty} \alpha_k^n y_k \leq \xi \tag{7.21}$$

and also

$$\gamma = \mathbf{E}[\lim_n g_n] \leq \lim_n \mathbf{E}[g_n] = \lim_n \gamma_n \leq \xi. \tag{7.22}$$

Moreover, since the function v is convex and bounded from below ($v \geq u(0)$ by (6.3)), we obtain

$$\mathbf{E}[v(g_n)] = \mathbf{E}\Big[v\Big(\lim_p \sum_{k=n}^{p} \alpha_k^n y_k Z^{(k)}\Big)\Big] \leq \sum_{k=n}^{\infty} \alpha_k^n \mathbf{E}[v(y_k Z^{(k)})], \tag{7.23}$$

once again by Fatou's lemma. Combining inequalities (7.21) and (7.23) we get

$$\begin{aligned} x\gamma_n + \mathbf{E}\Big[v\Big(\gamma_n \frac{d\mathbb{R}^n}{d\mathbb{P}}\Big)\Big] &\leq \sum_{k=n}^{\infty} \alpha_k^n (x y_k + \mathbf{E}[v(y_k Z^{(k)})]) \\ &= \sum_{k=n}^{\infty} \alpha_k^n w_k(x) \leq xy + \mathbf{E}[v(y\widehat{Z})] = w(x), \end{aligned} \tag{7.24}$$

where we have used representation (7.4) for the optimization problem in $[0,T]$ as well as for the ones in $[0,\tau_k]$, $k\in\mathbb{N}$. It is now easy to extend this formula from $\Big(\gamma_n,\dfrac{d\mathbb{R}^n}{d\mathbb{P}}\Big)$ to $\Big(\gamma,\dfrac{d\mathbb{R}}{d\mathbb{P}}\Big)$ in this way:

$$x\gamma+\mathbf{E}\Big[v\Big(\gamma\frac{d\mathbb{R}}{d\mathbb{P}}\Big)\Big]\le\lim_n\Big(x\gamma_n+\mathbf{E}\Big[v\Big(\gamma_n\frac{d\mathbb{R}^n}{d\mathbb{P}}\Big)\Big]\Big)\le w(x). \tag{7.25}$$

We use these inequalities to prove the following proposition, which is a fundamental step in the direction of our convergence results. Here we consider only the interesting case, i.e., the absolutely-continuous one.

Proposition 7.10. *Under the hypothesis of Theorem 7.9, the following assertions hold true:*

(i) The sequence $(g_n)_n$ is $\mathbb{P}$-uniformly integrable;

(ii) $\mathbb{R}^n\in\mathcal{M}^a(S^{\tau_n})$ and $\mathbb{R}\in\mathcal{M}^a(S^{\tau-})$;

(iii) $g=y\widehat{Z}=w'(x)\dfrac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}$.

Proof. (i) Recall that, under the Inada conditions, the RAE condition on the limit to $-\infty$ can be given in terms of the function v (see [69, Proposition 4.1]): there exist $\zeta_0>0$ and $C>0$ such that

$$\zeta v'(\zeta)\le Cv(\zeta),\quad \text{for}\,\zeta>\zeta_0. \tag{7.26}$$

Let us fix $K>0$ constant and consider the quantity $\mathbf{E}[g_n;g_n\ge K]$. If $K>\zeta_0$ and $v'(K)>0$, from (7.26) we get

$$g_n\le\frac{Cv(g_n)}{v'(g_n)}\le\frac{Cv(g_n)}{v'(K)},\quad\forall g_n\ge K,$$

where the last inequality holds because v' is increasing. In this case we have

$$\mathbf{E}[g_n;g_n\ge K]\le\frac{C}{v'(K)}\mathbf{E}[v(g_n);g_n\ge K],$$

and it is sufficient to prove the uniform boundedness of $\mathbf{E}[v(g_n);g_n\ge K],n\in\mathbb{N}$, to obtain the uniform integrability of $(g_n)_n$. Indeed, if

$$\mathbf{E}[v(g_n);g_n\ge K]<\eta\quad\forall n\in\mathbb{N},$$

for any $\epsilon > 0$ we clearly find a constant $K = K_\epsilon$ sufficiently large such that $\dfrac{C\eta}{v'(K)} < \epsilon$. Since the function v is continuous and strictly convex on $\mathbb{R}^+$ with $v(0) = u(\infty) < \infty$ (we are in the absolutely-continuous case), it is bounded on $[0, K]$. Therefore, to show that $(\mathbf{E}[v(g_n); g_n \geq K])_n$ is bounded or that $(\mathbf{E}[v(g_n)])_n$ is bounded, turns out to be equivalent. On the other hand, by (7.24) we have

$$\mathbf{E}[v(g_n)] = x\gamma_n + \mathbf{E}[v(g_n)] - x\gamma_n \leq w(x) - x\gamma_n$$

and, since $0 \leq \gamma_n \leq \xi < \infty$, the desired result follows.

(ii) Since S is assumed to be locally bounded, there exists $(\sigma_m)_{m\in\mathbb{N}}$ increasing sequence of stopping time such that $\sigma_m \uparrow \infty$ and $|S^{\sigma_m}| \leq C_m$ $\mathbb{P}$-a.s., for some C_m constant, $\forall m \in \mathbb{N}$. We now show that, $\forall\, n, m \in \mathbb{N}$, $S^{\sigma_m \wedge \tau_n}$ is a $\mathbb{R}^n$-martingale and $S^{\sigma_m\wedge\tau-}$ is a $\mathbb{R}$-martingale. Let us fix $n, m \in \mathbb{N}$ and $0 \leq s \leq t < T$ (or, eventually, permit $t = T$ if $T < \infty$). Since $(y_k Z^{(k)})_k$ is uniformly integrable from *(i)*, we have

$$\begin{aligned}
\mathbf{E}_{\mathbb{R}^n}[S_t^{\sigma_m\wedge\tau_n}|\mathcal{F}_s] &= \frac{\mathbf{E}[g_n S_t^{\sigma_m\wedge\tau_n}|\mathcal{F}_s]}{\mathbf{E}[g_n|\mathcal{F}_s]} = \frac{\mathbf{E}[\lim_p \sum_{k=n}^p \alpha_k^n y_k Z^{(k)} S_t^{\sigma_m\wedge\tau_n}|\mathcal{F}_s]}{\mathbf{E}[\lim_p \sum_{k=n}^p \alpha_k^n y_k Z^{(k)}|\mathcal{F}_s]} \\
&= \frac{\sum_{k=n}^\infty \alpha_k^n y_k \mathbf{E}[Z^{(k)} S_t^{\sigma_m\wedge\tau_n}|\mathcal{F}_s]}{\sum_{k=n}^\infty \alpha_k^n y_k \mathbf{E}[Z^{(k)}|\mathcal{F}_s]} = \frac{\sum_{k=n}^\infty \alpha_k^n y_k Z_s^{(k)} S_s^{\sigma_m\wedge\tau_n}}{\sum_{k=n}^\infty \alpha_k^n y_k Z_s^{(k)}} \\
&= S_s^{\sigma_m\wedge\tau_n},
\end{aligned}$$

by the fact that $S_t^{\sigma_m\wedge\tau_n}$ bounded implies $(y_k Z^{(k)} S_t^{\sigma_m\wedge\tau_n})_k$ and $(\sum_{k=n}^p \alpha_k^n y_k Z^{(k)} S_t^{\sigma_m\wedge\tau_n})_n$ uniformly integrable, for any $1 \leq n \leq p < \infty$ we fix. Here we have used the L^1-convergence of uniformly integrable sequences converging in probability and, in a similar way, we also obtain

$$\begin{aligned}
\mathbf{E}_{\mathbb{R}}[S_t^{\sigma_m\wedge\tau-}|\mathcal{F}_s] &= \frac{\mathbf{E}[g S_t^{\sigma_m\wedge\tau-}|\mathcal{F}_s]}{\mathbf{E}[g|\mathcal{F}_s]} = \frac{\mathbf{E}[\lim_n g_n S_t^{\sigma_m\wedge\tau_n}|\mathcal{F}_s]}{\mathbf{E}[\lim_n g_n|\mathcal{F}_s]} \\
&= \lim_n \frac{\mathbf{E}[g_n S_t^{\sigma_m\wedge\tau_n}|\mathcal{F}_s]}{\mathbf{E}[g_n|\mathcal{F}_s]} = S_s^{\sigma_m\wedge\tau-},
\end{aligned}$$

as claimed.

(iii) As was emphasized in Remark 7.7, the equivalence of the optimal problem in $[0, T]$ to the one in $[0, \tau[$ holds true and, in particular, $w_{\tau-}(x) = w(x)\ \ \forall\, x \in \mathbb{R}$. Hence, using *(ii)*

together with (7.25), the optimality of $\mathbb{R}$ immediately follows:

$$\mathbb{R} = \widehat{\mathbb{Q}}_y \quad \text{and} \quad \gamma = y = w'(x), \quad \text{so that} \quad g = y\widehat{Z}, \tag{7.27}$$

by formulation (7.8) of the original problem. What we have proved is that every convergent sequence $\left(\gamma_n \frac{d\mathbb{R}^n}{d\mathbb{P}}\right)_n$ of convex combinations of $\{y_n Z^{(n)}, n \in \mathbb{N}\}$, admits $y\widehat{Z}$ as limit. More precisely, by statement *(i)*, as $n \to \infty$ we have

$$\gamma_n \to y, \qquad \frac{d\mathbb{R}^n}{d\mathbb{P}} \xrightarrow{\mathbb{P}} \frac{d\widehat{\mathbb{Q}}_y}{d\mathbb{P}}$$

and

$$x\gamma_n + \mathbf{E}\Big[v\Big(\gamma_n \frac{d\mathbb{R}^n}{d\mathbb{P}}\Big)\Big] \to xy + \mathbf{E}[v(y\widehat{Z})] = w(x). \tag{7.28}$$

This proves the last assertion of the proposition and concludes the proof.

□

Proof. [Theorem 7.9] The first statement follows from (7.24) and (7.28), since $(w_n)_n$ is increasing and satisfies $w_n \leq w$.

(ii) It will first be shown that

$$y_n \frac{d\mathbb{Q}^{(n)}}{d\mathbb{P}} \xrightarrow{\mathbb{P}} y\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}.$$

In this purpose it is sufficient to prove that $(y_n Z^{(n)})_n$ is a sequence with the property to be "Cauchy in probability". We use the fact that the function v is strictly convex, hence uniformly strictly convex on compacts:
$\forall a > 0,\ K \in \mathbb{R}^+$ there exists $\beta > 0$ s.t. $\forall\,\delta_1,\ \delta_2$ with $\delta_1 \in [0, K]$ and $|\delta_1 - \delta_2| \geq a$, then

$$\frac{v(\delta_1) + v(\delta_2)}{2} > v\Big(\frac{\delta_1 + \delta_2}{2}\Big) + \beta. \tag{7.29}$$

Suppose that $(y_n Z^{(n)})_n$ is not Cauchy in probability, i.e., there exists $\alpha > 0$ s.t. $\forall N \in \mathbb{N}$ $\exists\, m = m_N, p = p_N > N$ with

$$\mathbb{P}\{|y_m Z^{(m)} - y_p Z^{(p)}| > \alpha\} > \alpha. \tag{7.30}$$

On the other hand, since $(y_n Z^{(n)})_n$ is uniformly integrable, there exists $K > 0$ such that

$$\mathbb{P}\{Z^{(n)} > K\} < \frac{\alpha}{2}, \quad \forall\, n \in \mathbb{N}.$$

Let us fix $N \in \mathbb{N}$ and $m, p > N$ satisfying (7.30), and define the sets

$$\tilde{\Omega} = \{\omega \in \Omega : |y_m Z^{(m)} - y_p Z^{(p)}| > \alpha\},\ \ \Omega_m = \{\omega \in \Omega : Z^{(m)} \le K\},\ \ \tilde{\Omega}_m = \tilde{\Omega} \cap \Omega_m.$$

It immediately follows that $\mathbb{P}(\tilde{\Omega}) > \alpha$, $\mathbb{P}(\Omega_m) \ge 1 - \alpha/2$ and $\mathbb{P}(\tilde{\Omega}_m) \ge \alpha/2$. Since in $\tilde{\Omega}_m$ (7.29) holds true, then we get

$$x\big(\frac{y_m + y_p}{2}\big) + \mathbf{E}\big[v\big(\frac{y_m Z^{(m)} + y_p Z^{(p)}}{2}\big)\big] \le$$

$$x\big(\frac{y_m + y_p}{2}\big) + \mathbf{E}\Big[\frac{v(y_m Z^{(m)}) + v(y_p Z^{(p)})}{2} 1_{\tilde{\Omega}_m^c}\Big] + \mathbf{E}\Big[\big(\frac{v(y_m Z^{(m)}) + v(y_p Z^{(p)})}{2} - \beta\big) 1_{\tilde{\Omega}_m}\Big]$$

$$= x\big(\frac{y_m + y_p}{2}\big) + \mathbf{E}\Big[\frac{v(y_m Z^{(m)}) + v(y_p Z^{(p)})}{2}\Big] - \beta \mathbb{P}(\tilde{\Omega}_m)$$

$$\le \frac{1}{2}\big[x y_n + \mathbf{E}[v(y_n Z^{(n)})] + x y_p + \mathbf{E}[v(y_p Z^{(p)})]\big] - \beta\frac{\alpha}{2}.$$

Hence, putting

$$\eta_N = \frac{y_m + y_p}{2}, \quad \eta_N \frac{d\mathbb{M}^N}{d\mathbb{P}} = \frac{y_m Z^{(m)} + y_p Z^{(p)}}{2},$$

we have

$$\limsup_{N \to \infty} \Big\{x\eta_N + \mathbf{E}\Big[v\Big(\eta_N \frac{d\mathbb{M}^N}{d\mathbb{P}}\Big)\Big]\Big\} \le xy + \mathbf{E}[v(y\widehat{Z})] - \beta\frac{\alpha}{2}.$$

Now, by possibly passing to a convergent sequence $\Big(\bar{\gamma}_k \frac{d\bar{\mathbb{Q}}^k}{d\mathbb{P}}\Big)_k$ of convex combinations of $\Big\{\eta_N \frac{d\mathbb{M}^N}{d\mathbb{P}},\ N \in \mathbb{N}\Big\}$ (if $\big(\eta_N \frac{d\mathbb{M}^N}{d\mathbb{P}}\big)_{N \in \mathbb{N}}$ results not to be convergent), for any k in $\mathbb{N}$ we get

$$x\bar{\gamma}_k + \mathbf{E}\Big[v\Big(\bar{\gamma}_k \frac{d\bar{\mathbb{Q}}^k}{d\mathbb{P}}\Big)\Big] \le xy + \mathbf{E}[v(y\widehat{Z})] - \beta\frac{\alpha}{2} < w(x),$$

with the same arguments used to obtain (7.24). On the other hand, by (7.28) we have

$$x\bar{\gamma}_k + \mathbf{E}\Big[v\Big(\bar{\gamma}_k \frac{d\bar{\mathbb{Q}}^k}{d\mathbb{P}}\Big)\Big] \underset{k}{\to} w(x),$$

in contradiction with the preceding inequalities. This proves that $(y_n Z^{(n)})_n$ is Cauchy in probability and therefore it also converges in probability. From the uniform integrability, this limit also holds in the $L^1(\mathbb{P})$-sense and, by Proposition 7.10, it equals $y\widehat{Z}$. What we have shown is the convergence

$$y_n Z^{(n)} \overset{L^1(\mathbb{P})}{\longrightarrow} y\widehat{Z}, \quad \text{or} \quad Z^{(n)} \overset{L^1(\mathbb{P})}{\longrightarrow} \widehat{Z},$$

and by (7.2) and (7.17) we also have

$$X_T^{(n)}(x) \xrightarrow{\mathbb{P}} \widehat{X}_T(x),$$

as claimed.

□

As noted before, plugging (7.18) into Theorem 7.9 we obtain an approximation of the optimal terminal wealth $\widehat{X}_T(x)$ via self-financing trading strategies. Therefore, in the absolutely-continuous case, an economic agent can realize a wealth as large as she wants with strictly positive probability (greater than or equal to $\mathbb{P}(A)$, with A defined in Remark 6.5).

Chapter 8

Absolutely-Continuous Case: An Example

In this chapter we construct a class of examples which show how the absolutely-continuous case may occur for any utility function fulfilling our requests.

Let $u : \mathbb{R} \to \mathbb{R}$ be a utility function satisfying Assumption 6.1. We construct a real-valued (locally) bounded semimartingale $S = (S_n)_{n \in \mathbb{N}_0}$ based on and adapted to a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in \mathbb{N}_0}, \mathbb{P})$, which describes the discounted price process of a risky traded asset. Since we are in discrete time, Assumption 7.4 holds automatically true. We shall give conditions on $\mathbb{P}$ and u in order to make the financial market modelled by S satisfying Assumptions 6.2, 6.3 and fitting in with the absolutely-continuous case. This means that, for an economic agent investing in this market, the optimization problem (6.4) produces a martingale measure lying in $\mathcal{M}^a(S)$ but not in $\mathcal{M}^e(S)$. Moreover, the optimal terminal wealth will have the nice representation $(\widehat{H} \cdot S)_\infty = (1 \cdot S)_\infty = \lim_{n \to \infty} S_n$ where this limit exists, otherwise it will be equal to $+\infty$.

8.1 The Market Model

To simplify the notation we fix the initial endowment at $x = 0$, so that the primal problem takes the form

$$w(0) = \sup_{H \in \mathcal{H}} \mathbf{E}[u((H \cdot S)_\infty)]. \tag{8.1}$$

In order to define the asset price process S, we choose a trinomial tree model in which, at every step n, one can "go up", "go down" or remain at the same level. To this end, we consider two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ of real numbers such that $a_1 = 0$, $\text{sgn}(a_n) = \text{sgn}(b_n) = (-1)^n$ $\forall n \in \mathbb{N}$, $|b_n| > |a_n|$ and $|a_n|$ (so $|b_n|$ too) increases to $+\infty$ as $n \to +\infty$. Let us put $S_0 = 0$ in $\Omega =: C_0$, then split C_0 in tree sets, say A_1 (where S remains at the same level 0), B_1 (where the process goes down at b_1) and C_1 (where it goes up at a_2). In the same way, for any n in $\mathbb{N}$, we put $S_n = S_{n-1}$ in $\bigcup_{k=1}^{n-1}(A_k \cup B_k)$, whereas C_{n-1} is split into three sets, say A_n, B_n and C_n, where we define S_n equal to a_n, b_n and a_{n+1}, respectively. Therefore, at time n with n even (resp. odd), the process can go up, if B_n (resp. C_n) occurs, it can go down, if C_n (resp. B_n) occurs, or remain at the same level, in the sets $A_1, \ldots, A_n, B_1, \ldots, B_{n-1}$. In this way the process we have constructed works as follows, for any $n \in \mathbb{N}$:

$$S_n = \begin{cases} a_k, & \text{on } A_k,\ k = 1, \ldots, n, \\ b_k, & \text{on } B_k,\ k = 1, \ldots, n, \\ a_{n+1}, & \text{on } C_n. \end{cases}$$

Moreover, as n goes to $+\infty$, S admits path-wise limit on $\bigcup_{n=1}^{\infty}(A_n \cup B_n)$, whereas it oscillates between $+\infty$ and $-\infty$ on $C_\infty = \bigcap_{n=0}^{\infty} C_n$. Since we will put C_∞ not null under the historical probability, the limit of the process is not almost surely well defined and it becomes convenient to introduce the random variable

$$S_\infty = \lim_n S_n 1_{C_\infty^c}.$$

This construction follows an analogue pattern such as the one in [70] and here we can give a similar interpretation of the asset price process. Indeed, we may consider S as the value of a player suitably-stopped portfolio, when the game consists of a sequence of independent experiments with the three outcomes:

$$\begin{cases} u_n : & \text{to go up in the } n^{th} \text{ trial}, \\ m_n : & \text{to remain at the same level in the } n^{th} \text{ trial}, \\ d_n : & \text{to go down in the } n^{th} \text{ trial}. \end{cases}$$

The value of the game at $n = 0$ is fixed equal to zero, and the increments we consider are:

$$\eta_n = \begin{cases} a_{n+1} - a_n, & \text{if } u_n \text{ and } n \text{ odd, or } d_n \text{ and } n \text{ even, occurs,} \\ 0, & \text{if } m_n \text{ occurs,} \\ b_n - a_n, & \text{if } d_n \text{ and } n \text{ odd, or } u_n \text{ and } n \text{ even, occurs.} \end{cases}$$

Hence, as long as the player continues the game, his portfolio value at time $n \in \mathbb{N}$ is

$$M_n = \sum_{k=1}^{n} \eta_k.$$

Now the idea is to play as long as experiments have outcomes of up-type when n odd and of down-type when n even, while the game stops at the first time in which this does not occur. Let us introduce the stopping times

$$\theta := \inf\{n : m_n \text{ occurs}\}, \sigma := \inf\{n : d_n \text{ and } n \text{ odd, or } u_n \text{ and } n \text{ even, occurs}\}, \rho := \theta \wedge \sigma.$$

If we define our sets as follows:

$$A_n = \{n = \theta < \sigma\},\ \ B_n = \{n = \sigma < \theta\},\ \ C_n = \{\rho > n\} \quad \text{and} \quad C_\infty = \{\rho = \infty\},$$

and allow the gambler to play up to the random time ρ, his portfolio value turns out to be modelled by the stopped process

$$M_n^\rho = S_n.$$

After this comment, let us come back to the definition of our model and denote by $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$ the natural filtration generated by S:

$\mathcal{F}_0 = \{\emptyset, \Omega\}$ is the trivial algebra,

$\mathcal{F}_n = \sigma(S_n) = \sigma(\{A_1, \ldots, A_n, B_1, \ldots, B_n, C_n\}), \forall n \in \mathbb{N}$ and

$\mathcal{F} = \mathcal{F}_\infty = \bigvee_n \mathcal{F}_n = \sigma(\{(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}}, C_\infty\})$.

8.2 Probability Measures

8.2.1 Characterization of the Martingale Measures

The unique condition needed on the sequences $(a_n)_n$ and $(b_n)_n$ will be obtained from the characterization of the martingale measures for the process S (and independently of the utility function u we consider). Therefore, before we introduce the assumption on the historical probability $\mathbb{P}$, we proceed to identify the set $\mathcal{M}^a(S)$. Of course every measure $\mathbb{Q} \in \mathcal{M}^a(S)$ has to satisfy the martingale property

$$\mathbb{Q}(B_n)(b_n - a_n) + \mathbb{Q}(C_n)(a_{n+1} - a_n) = 0, \quad \forall n \in \mathbb{N}, \tag{8.2}$$

that is

$$\frac{\mathbb{Q}(B_n)}{\mathbb{Q}(C_n)} = \frac{a_n - a_{n+1}}{b_n - a_n} =: \xi_n, \quad \forall n \in \mathbb{N}. \tag{8.3}$$

To have $\mathcal{M}^e(S) \neq \emptyset$, we need a probability measure $\mathbb{Q}$ satisfying (8.3) and such that $\mathbb{Q} \sim \mathbb{P}$. This measure clearly satisfies $\lim_n \mathbb{Q}(B_n) = 0$ and, requiring $\mathbb{P}(C_\infty) > 0$, also $\lim_n \mathbb{Q}(C_n) > 0$, so that $\xi_n \underset{n}{\to} 0$. More precisely, at every step n we can rewrite:

$$\mathbb{Q}(B_n) = \xi_n \mathbb{Q}(C_n) \quad \text{and} \quad \mathbb{Q}(A_n) = \gamma_n \mathbb{Q}(C_n)$$

for some γ_n, so that $\mathbb{Q}(C_{n-1}) = (1 + \gamma_n + \xi_n)\mathbb{Q}(C_n)$. Now, since we want

$$0 < \mathbb{Q}(C_\infty) = \lim_n \mathbb{Q}(C_n) = \lim_n \prod_{k=1}^{n} \frac{1}{1 + \gamma_k + \xi_k} = \prod_{n=1}^{\infty} \frac{1}{1 + \gamma_n + \xi_n},$$

we need the following condition to hold:

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{1 + \gamma_n + \xi_n}\right) < \infty,$$

or equivalently:

$$\frac{\gamma_n + \xi_n}{1 + \gamma_n + \xi_n} \downdownarrows 0.$$

Here, given two sequences $(f_n)_n \subset \mathbb{R}$ and $(g_n)_n \subset \mathbb{R} \setminus \{0\}$, by the notation $\frac{f_n}{g_n} \downdownarrows 0$ or $f_n \ll g_n$ we mean that $\sum_{n=1}^{\infty} \frac{f_n}{g_n} < \infty$, whereas by $f_n \approx g_n$ we indicate that $\frac{f_n}{g_n} \in [c^{-1}, c]$, asymptotically, for some $c > 1$. Arranging things such that $\mathbb{Q}(A_n)$ tends to zero sufficiently quickly ($\gamma_n \ll \xi_n$), we can relax the last requirement to the following one:

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{1 + \xi_n}\right) = \sum_{n=1}^{\infty} \frac{a_n - a_{n+1}}{b_n - a_{n+1}} < \infty. \tag{8.4}$$

We then obtain that the corresponding martingale measure $\mathbb{Q}$ lies in $\mathcal{M}^e(S)$.

Note that, in order to have (8.4) satisfied, it is sufficient to take

$$|a_{n+1}| \ll |b_n|, \tag{8.5}$$

so we may and do assume this condition holds true. In this way Assumption 6.2 turns out to be satisfied. Requirement (8.5) is the only one we make on the sequences $(a_n)_n$ and $(b_n)_n$, and we point out that it does not depend on the utility function u describing the agent preferences. To give an example, a good choice for these parameters is

$$a_n \approx (-1)^n n \quad \text{and} \quad b_n \approx (-2)^n.$$

8.2.2 The Historical Probability

We now put some conditions on the historical probability $\mathbb{P}$, to make the solution of (8.1) having the announced representation. What we want is the optimal wealth to be equal to $(\widehat{H} \cdot S)_\infty = (1 \cdot S)_\infty = S_\infty$ on C_∞^c, and to $+\infty$ on C_∞ (where $\mathbb{P}(C_\infty) > 0$). This result motivates the following requirements:

(P1) $\mathbb{P}(A_n)u'(a_n) \downdownarrows 0$ and $\mathbb{P}(B_n)u'(b_n) \downdownarrows 0$;

(P2) $\mathbb{P}(A_{n+1})u'(a_{n+1}) \approx \sum_{k=n+1}^{\infty} \big(\mathbb{P}(A_k)u'(a_k) + \mathbb{P}(B_k)u'(b_k)\big)$;

(P3) $\mathbb{P}(B_n)u'(b_n) = \xi_n \sum_{k=n+1}^{\infty} \big(\mathbb{P}(A_k)u'(a_k) + \mathbb{P}(B_k)u'(b_k)\big)$,
so that, if (P2) holds, $\mathbb{P}(B_n)u'(b_n) \approx \xi_n \mathbb{P}(A_{n+1})u'(a_{n+1})$;

(P4) $|a_n|\mathbb{P}(A_n)u'(a_n) \downdownarrows 0$,
which, if (P2) and (P3) hold, implies $|b_n|\mathbb{P}(B_n)u'(b_n) \downdownarrows 0$, by (8.5);

(P5) $\mathbb{P}(A_n)u(-|a_n|) \downdownarrows 0$ and $\mathbb{P}(B_n)u(-|b_n|) \downdownarrows 0$;

(P6) $\sum_{n=1}^{\infty} \big(\mathbb{P}(A_n) + \mathbb{P}(B_n)\big) < 1$ (i.e. $\mathbb{P}(C_\infty) > 0$, as mentioned earlier).

Note that, since $|a_n|$ and $|b_n|$ increase to $+\infty$, (P4) is clearly stronger than (P1) and therefore the requirements we make on $\mathbb{P}$ are just (P2)$-$(P6). In the next subsection it will become clear why we require these conditions. Here we only give an example in the significant case of the exponential utility.

Example 8.1. *Consider the preferences of the agent as modeled by the exponential utility function $u(x) = -\exp(-x)$ and, as before, put $a_n \approx (-1)^n n$ and $b_n \approx (-2)^n$. In this setting, a good choice for the measure $\mathbb{P}$ is*

$$\mathbb{P}(A_n) \approx \exp(-2^n + a_n) \quad \textit{and} \quad \mathbb{P}(B_n) \approx n2^{-n} \exp(-2^{n+1} + b_n),$$

which makes (P2)–(P6) satisfied.

8.2.3 The Minimax Martingale Measure

Let us define a measure $\widetilde{\mathbb{Q}}$ absolutely continuous with respect to $\mathbb{P}$, in the following way:

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} = \frac{u'(X_\infty)}{c}, \qquad \text{where} \quad X_\infty = \begin{cases} S_\infty, & \text{on } C_\infty^c \\ +\infty, & \text{on } C_\infty \end{cases} \tag{8.6}$$

and $c = \|u'(X_\infty)\|_{L^1(\mathbb{P})} < \infty$ by (P1). We obtain

$$\widetilde{\mathbb{Q}}(A_n) = c^{-1}u'(a_n)\mathbb{P}(A_n), \ \widetilde{\mathbb{Q}}(B_n) = c^{-1}u'(b_n)\mathbb{P}(B_n) \ \text{ and } \ \widetilde{\mathbb{Q}}(C_\infty) = c^{-1}u'(\infty)\mathbb{P}(C_\infty) = 0,$$

so that, in particular, $\widetilde{\mathbb{Q}}$ is not equivalent to $\mathbb{P}$. Moreover, by (P3),

$$\widetilde{\mathbb{Q}}(B_n) = \xi_n \sum_{k=n+1}^{\infty} \big(\widetilde{\mathbb{Q}}(A_k) + \widetilde{\mathbb{Q}}(B_k)\big) = \xi_n \widetilde{\mathbb{Q}}(C_n), \tag{8.7}$$

i.e., $\widetilde{\mathbb{Q}}$ satisfies the martingale property (8.3). It follows that $\widetilde{\mathbb{Q}}$ is a good candidate to be the minimax martingale measure $\widehat{\mathbb{Q}}_c$ and we will show that, under our assumptions, this will be the case. Before proving the optimality of $\widetilde{\mathbb{Q}}$, we show the following properties:

(i) $X_\infty \in L^1(\widetilde{\mathbb{Q}})$ with $\mathbf{E}_{\widetilde{\mathbb{Q}}}[X_\infty] = 0$;

(ii) $X_\infty \in L^1(\mathbb{Q})$ with $\mathbf{E}_{\mathbb{Q}}[X_\infty] = 0, \quad \forall \mathbb{Q} \in \mathcal{M}_f^a(S)$.

By (8.6) and (P4) we clearly have

$$\mathbf{E}_{\widetilde{\mathbb{Q}}}[|X_\infty|] = \mathbf{E}_{\widetilde{\mathbb{Q}}}[|S_\infty|] = \sum_{n=1}^{\infty} \big(|a_n|\widetilde{\mathbb{Q}}(A_n) + |b_n|\widetilde{\mathbb{Q}}(B_n)\big) < \infty$$

and also

$$\mathbf{E}_{\widetilde{\mathbb{Q}}}[|X_\infty - S_n|] \underset{n\to\infty}{\longrightarrow} 0, \quad \text{i.e.} \quad S_n \overset{L^1(\widetilde{\mathbb{Q}})}{\longrightarrow} X_\infty,$$

by approximation in (P2). This yields $(S_n)_n$ uniformly integrable with respect to $\widetilde{\mathbb{Q}}$, with

$$\mathbf{E}_{\widetilde{\mathbb{Q}}}[X_\infty|\mathcal{F}_n] = \mathbf{E}_{\widetilde{\mathbb{Q}}}[S_\infty|\mathcal{F}_n] = S_n \quad \text{and} \quad \mathbf{E}_{\widetilde{\mathbb{Q}}}[X_\infty] = \mathbf{E}_{\widetilde{\mathbb{Q}}}[S_\infty] = 0, \tag{8.8}$$

as claimed. Let now $\mathbb{Q}$ be any measure in $\mathcal{M}^a_f(S)$. Since

$$\mathbf{E}\Big[v\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big] = \sum_{n=1}^{\infty}\Big\{v\Big(\frac{\mathbb{Q}(A_n)}{\mathbb{P}(A_n)}\Big)\mathbb{P}(A_n) + v\Big(\frac{\mathbb{Q}(B_n)}{\mathbb{P}(B_n)}\Big)\mathbb{P}(B_n)\Big\} + v\Big(\frac{\mathbb{Q}(C_\infty)}{\mathbb{P}(C_\infty)}\Big)\mathbb{P}(C_\infty)$$

is a finite quantity and the function v is bounded from below, we have

$$v\Big(\frac{\mathbb{Q}(A_n)}{\mathbb{P}(A_n)}\Big)\mathbb{P}(A_n) \downdownarrows 0 \quad \text{and} \quad v\Big(\frac{\mathbb{Q}(B_n)}{\mathbb{P}(B_n)}\Big)\mathbb{P}(B_n) \downdownarrows 0.$$

Therefore, using (6.3) for $y = \dfrac{d\mathbb{Q}}{d\mathbb{P}}$ and $x_n = -|a_n|$ or $-|b_n|$, we obtain

$$|a_n|\mathbb{Q}(A_n) \leq v\Big(\frac{\mathbb{Q}(A_n)}{\mathbb{P}(A_n)}\Big)\mathbb{P}(A_n) - u(-|a_n|)\mathbb{P}(A_n) \downdownarrows 0$$

and

$$|b_n|\mathbb{Q}(B_n) \leq v\Big(\frac{\mathbb{Q}(B_n)}{\mathbb{P}(B_n)}\Big)\mathbb{P}(B_n) - u(-|b_n|)\mathbb{P}(B_n) \downdownarrows 0,$$

by assumption (P5). This yields the integrability of X_∞ with respect to every $\mathbb{Q}$ in $\mathcal{M}^a_f(S)$, with $\mathbf{E}_{\mathbb{Q}}[X_\infty] = \mathbf{E}_{\mathbb{Q}}[S_\infty] = 0$, and in particular $\mathbb{Q}(C_\infty) = 0$ (so that $\mathbb{Q} \ll \widetilde{\mathbb{Q}}$). Now we are able to prove the optimality of the measure $\widetilde{\mathbb{Q}}$ in the set $\mathcal{M}^a(S)$. Indeed, for any given probability measure $\mathbb{Q} \in \mathcal{M}^a_f(S)$, we have

$$\begin{aligned}
\mathbf{E}\Big[v\Big(c\frac{d\mathbb{Q}}{d\mathbb{P}}\Big) - v\Big(c\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\Big)\Big] &\geq \mathbf{E}\Big[v'\Big(c\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\Big)\Big(c\frac{d\mathbb{Q}}{d\mathbb{P}} - c\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\Big)\Big] \\
&= c\mathbf{E}_{\mathbb{Q}}\Big[-(u')^{-1}\Big(c\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\Big)\Big] - c\mathbf{E}_{\widetilde{\mathbb{Q}}}\Big[-(u')^{-1}\Big(c\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\Big)\Big] \\
&= c\mathbf{E}_{\mathbb{Q}}[-X_\infty] - c\mathbf{E}_{\widetilde{\mathbb{Q}}}[-X_\infty] = 0,
\end{aligned}$$

by the convexity of v. Consequently $\widetilde{\mathbb{Q}}$ turns out to be the optimal solution $\widehat{\mathbb{Q}}_c$ to the dual problem. On the other hand, by (7.3) we have $w'(0) = (-\nu')^{-1}(0) = c$. Then the measure $\widetilde{\mathbb{Q}} \in \mathcal{M}^a(S)$ we have defined is the optimal martingale measure $\widehat{\mathbb{Q}}_{w'(0)}$ and its density is proportional to the marginal utility of X_∞. Moreover, by (8.6) and (P6) we know that this martingale measure is not equivalent to the probability measure $\mathbb{P}$. By this fact and guided by relation (7.2), we obtain that the optimal terminal wealth $\widehat{X}_\infty(0)$ equals X_∞, which is infinite with strictly positive probability by (P6). Then (8.1) produces $w(0) = \mathbf{E}[u(X_\infty)] < \infty$ by assumption (P5).

8.3 On the Approximation of the Optimal Wealth

We can observe that a sequence $(K^m)_{m\in\mathbb{N}}$ of admissible trading strategies permitting the approximation of the optimal wealth, is given by $K^m = 1_{[0,2m-1]}$. In this way, we obtain processes $Y^m := (K^m \cdot S)$ such that

$$Y^m_\infty = (K^m \cdot S)_{2m-1} = S_{2m-1} \underset{m}{\rightarrow} \widehat{X}_\infty(0)$$

and also

$$\mathbf{E}[u(Y^m_\infty)] \underset{m}{\rightarrow} \mathbf{E}[u(\widehat{X}_\infty(0))] = w(0).$$

On the contrary, we can easily see that

$$\widehat{X}_n(0) := \mathbf{E}_{\widehat{\mathbb{Q}}}[\widehat{X}_\infty(0)|\mathcal{F}_n] = S_n \underset{n}{\nrightarrow} \widehat{X}_\infty(0).$$

Also regarding the auxiliary optimization problems introduced in Section 7.1, we get

$$X^{(n)}_\infty(0) = (H^n \cdot S)_{\tau_n} \underset{n}{\rightarrow} \widehat{X}_\infty(0),$$

by our main theorem, but

$$\widehat{X}_{\tau_n}(0) := \mathbf{E}_{\widehat{\mathbb{Q}}}[\widehat{X}_\infty(0)|\mathcal{F}_{\tau_n}] \underset{n}{\nrightarrow} \widehat{X}_\infty(0).$$

Let us consider the initial interpretation of S as a player portfolio value, in order to reinterpret the processes Y^m now defined. For any n in $\mathbb{N}$, we have

$$Y^m_n = S_{n\wedge(2m-1)} = M^{\rho\wedge\rho_m}_n,$$

where ρ_m is the deterministic stopping time $\rho_m = 2m - 1$. Hence, we can approximate the optimal terminal wealth through portfolio values of players which stop at time $\rho \wedge \rho_m$ the game described at the beginning. In other words, playing this game (i.e., trading in our market S) we can obtain a wealth that with strictly positive probability ($> \mathbb{P}(C_\infty)$) is as large as we want.

Remark 8.2. *About the choice of the time horizon $+\infty$, we point out that it is not relevant. To have the same examples in a finite trading horizon T, it is sufficient to consider the time scale $T\frac{n}{n+1}$ instead of n.*

Remark 8.3. *Regarding the unboundedness of the asset price process, we can see how to drop it, leaving unchanged the optimal wealth process* $\widehat{X}(0)$*. Let us define a new process* $S^b = (S^b_n)_{n\in\mathbb{N}}$ *as follows:*

$$S^b_0 = 0 \quad \textit{and} \quad S^b_n - S^b_{n-1} = \Delta S^b_n = c_n \Delta \widehat{X}_n(0) = c_n \Delta S_n.$$

Of course, for arbitrary constants $c_n \neq 0$*, the optimal wealth process obtained by trading in the market* S^b *is the same as the one obtained by trading in* S*. So we can choose some constants* c_n *such that* S^b *is a uniformly bounded process, for example* $c_n = \frac{2^{-(n+1)}}{|b_{n+1}|}$*, which gives us* $|S^b_n| \leq 1 - 2^{-n}$*,* $\forall n \in \mathbb{N}$*, and also* $|S^b_\infty| \leq 1$*.*

CONCLUSIONS

For any utility function satisfying our assumptions, we have shown how the absolutely-continuous case may occur. In such a situation, we lose the representability of the optimal wealth $\widehat{X}_T$ as the final value of a trading strategy. However, since $\widehat{X}_T(x)$ solves (6.1), we know that there exists a sequence $(K^n)_{n\in\mathbb{N}}$ of admissible strategies such that $\mathbf{E}[u(x + (K^n \cdot S)_T)]$ converges to $\mathbf{E}[u(\widehat{X}_T(x))]$. In addition to this, by solving problems (7.15) we get a sequence $(H^n)_n$ of self-financing strategies, characterized in (7.18), which are not necessarily admissible, but such that

$$u(x + (H^n \cdot S)_T) \xrightarrow{\mathbb{P}} u(\widehat{X}_T(x)),$$

by Theorem 7.9. This result taken in conjunction with representation (7.18) produces

$$\widehat{X}_T(x) = x + \lim_{n\to\infty} (H^n \cdot S)_{\tau_n}, \tag{8.9}$$

when the limit is taken in probability. Therefore, although the optimal terminal wealth is not perfectly replicable, we obtain it as the limit of portfolio values attainable by trading in the market.

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