Absolutely continuous optimal martingale measures

Beatrice Acciaio

Summary: In this paper, we consider the problem of maximizing the expected utility of terminal wealth in the framework of incomplete financial markets. In particular, we analyze the case where an economic agent, who aims at such an optimization, achieves infinite wealth with strictly positive probability. By convex duality theory, this is shown to be equivalent to having the minimal-entropy martingale measure $\hat{\mathbf{Q}}$ non-equivalent to the historical probability $\mathbf{P}$ (what we call the *absolutely-continuous case*). In this anomalous case, we no longer have the representation of the optimal wealth as the terminal value of a stochastic integral, stated in Schachermayer[9] for the case of $\hat{\mathbf{Q}} \sim \mathbf{P}$ (i.e. the *equivalent case*). Nevertheless, we give an approximation of this terminal wealth through solutions to suitably-stopped problems, solutions which still admit the integral representation introduced in [9]. We also provide a class of examples fitting to the absolutely-continuous case.

1 Introduction

A subject of great importance in Mathematical Finance is the problem of an economic agent who trades in a financial market so as to maximize the expected utility of her terminal wealth. Let $T \in (0, +\infty]$ be the time horizon and $S = (S_t)_{0 \leq t \leq T}$ be an $\mathbb{R}^d$-valued locally-bounded semimartingale based on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$, modelling the discounted price process of d tradeable assets. In general, given $x \in \mathbb{R}$ the initial endowment, this optimization problem can be written as:

$$u(x) = \sup_H \mathbf{E}[U(x + (H \cdot S)_T)], \tag{1}$$

where the utility function U describes agent preferences and H runs through a suitable set of admissible trading strategies (depending on the process S we consider). The classical assumptions made on the function U are smoothness (i.e. U continuously differentiable) and monotonicity -of both the utility and the marginal utility- (i.e. U strictly increasing and U' strictly decreasing), whereas, in order to rule out doubling strategies, the concept of admissibility is usually given as follows: the trading rules H are predictable and S-integrable processes for which the stochastic integral w.r. to S is uniformly bounded from below.

In this paper we study the maximization problem in an incomplete market where we allow the wealth processes to be negative, so that the utility functions we consider are defined and finitely valued on the entire real line. In this setting, basic results of existence and uniqueness of the optimal solution are given in Schachermayer[9], to which we refer for any unexplained notation and for a complete outline of the situation. In order to apply these well-known results, on the market S and on the utility function U we make the assumptions introduced below.

Let us denote by $\mathcal{M}^a(S)$ (resp. $\mathcal{M}^e(S)$) the set of absolutely continuous (resp. equivalent) local martingale measures: probability measures $\mathbf{Q} \ll \mathbf{P}$ (resp. $\mathbf{Q} \sim \mathbf{P}$) such that S is a $\mathbf{Q}$-local martingale. Throughout the paper we shall assume the following formulation of the no arbitrage condition:

Assumption 1.1 *The set $\mathcal{M}^e(S)$ is not empty.*

This condition insures that in the market modelled by S there are *no* possibilities of *free lunch with vanishing risk* (NFLVR) (for this version of the Fundamental Theorem of Asset Pricing, see [4]).

On the other hand, the utility function U is assumed to behave according to some technical requirements:

Assumption 1.2 *The utility function $U : \mathbb{R} \to \mathbb{R}$ is smooth, strictly increasing, strictly concave and satisfies the following conditions:*

(I) Inada conditions: $\lim_{x\to-\infty} U'(x) = \infty$ *and* $\lim_{x\to+\infty} U'(x) = 0$*;*

(II) Reasonable Asymptotic Elasticity (RAE) condition (see [9] for the significance): $\liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1$ *and* $\limsup_{x\to+\infty} \frac{xU'(x)}{U(x)} < 1.$

Furthermore, to exclude the trivial degenerate case, we make the following intuitive assumption involving both U and S:

Assumption 1.3 *For any stopping time $\rho \in [0,T]$,*

$$\sup_{H=H1_{]\rho,T]}} \boldsymbol{E}[U((H\cdot S)_T)|\mathcal{F}_\rho] < U(\infty) \text{ a.s.} \tag{2}$$

In particular, this serves to insure the finiteness of u on all of $\mathbb{R}$.

In [9] the use of the powerful tools of duality theory (see e.g. [7, 8, 1, 10, 2], too, for this kind of approach to optimization problems) leads to the characterization of the maximizer to (1) in terms of the optimal solution to a dual variational problem. We, too, will use this type of approach and, in order to formalize it, we need to introduce the convex conjugate $V : \mathbb{R}_+ \to \mathbb{R}$ of the utility function U:

$$V(y) = \sup_{x\in\mathbb{R}}(U(x) - xy), \quad \forall y > 0. \tag{3}$$

Under our assumptions, V turns out to be a smooth and strictly convex function such that $V(0) = U(+\infty)$, $V(+\infty) = +\infty$ and $V'(0) = -\infty$, $V'(+\infty) = +\infty$. The optimal problem dual to (1) can now be expressed in the following way:

$$v(y) = \inf_{\mathbf{Q}\in\mathcal{M}^a(S)} \mathbf{E}\Big[V\Big(y\frac{d\mathbf{Q}}{d\mathbf{P}}\Big)\Big], \tag{4}$$

where the function $v : \mathbb{R}_+ \to \mathbb{R}$ is finite from (2). Moreover, in the context of the present paper, we can apply a result of Bellini and Frittelli[2] which implies that the unique minimizer of (4) (the so-called minimax measure) exists and is in fact in $\mathcal{M}^a(S)$. Now, the basic idea of the dual approach is to solve the latter problem and then, by convex duality, to solve the former one. Define $\hat{X}_T(x)$ as the optimal terminal wealth, solution to the primal problem (1), and $\hat{\mathbf{Q}}_y$ as the minimal martingale measure, solution to the dual problem (4). The crucial formula which relates these optimizers is given by

$$\frac{d\hat{\mathbf{Q}}_y}{d\mathbf{P}} = \frac{U'(\hat{X}_T(x))}{y} \tag{5}$$

where $y = u'(x) > 0$ (see Theorem 2.1 below).

Let us now outline the direction this paper moves in. To this end, we identify two mutually exclusive situations, in terms of the optimal martingale measure:

Equivalent case: $\hat{\mathbf{Q}} \sim \mathbf{P}$ (i.e. $\hat{\mathbf{Q}} \in \mathcal{M}^e(S)$);
Absolutely-Continuous case: $\hat{\mathbf{Q}} \ll \mathbf{P}, \mathbf{P} \not\ll \hat{\mathbf{Q}}$ (i.e. $\hat{\mathbf{Q}} \in \mathcal{M}^a(S) \setminus \mathcal{M}^e(S)$).

Here and throughout the paper, where it does not generate confusion, we do not indicate the dependence -of the optimal solutions- on the initial capital x that the agent is endowed with.

In the equivalent case, the solution to the primal problem is shown to be equal to the final wealth of some self-financing strategy (Theorem 2.1 below). More precisely, the optimal wealth at the time horizon T can be obtained as the terminal value of a $\hat{\mathbf{Q}}_{u'(x)}$-uniformly integrable martingale: $\hat{X}_T(x) = x + (\hat{H}(x) \cdot S)_T$. If, on the contrary, the minimax measure $\hat{\mathbf{Q}}$ is just absolutely continuous with respect to $\mathbf{P}$, we lose this characterization, in the sense that the integral representation of $\hat{X}_T$ remains valid under the optimal measure only (i.e. $\hat{\mathbf{Q}} - a.s.$). In this paper we focus on the absolutely-continuous case and show how to reach the wealth optimal at time T by means of news problems, fitting to the equivalent case, which are defined in some random intervals contained in $[0, T]$.

Let $A \in \mathcal{F}$ denote the maximal set such that $\hat{\mathbf{Q}}(A) = 0$. By relation (5) and the Inada conditions, we clearly have $A = \big\{\frac{d\hat{\mathbf{Q}}}{d\mathbf{P}} = 0\big\} = \big\{\hat{X}_T = +\infty\big\}$ $\mathbf{P}$-almost surely. On the other hand, the absolutely-continuous case results in $\mathbf{P}(A) > 0$. Then, in this setting, an economic agent trading in the market can realize a wealth as large as she wants with a strictly positive probability (greater than or equal to $\mathbf{P}(A)$). Despite a -maybe sceptical- first impression one may get about the occurrence of this phenomenon, in Section 3 we show that this case may arise for any utility function we consider.

The main goal of this paper is to approximate the optimal solution to the problem (1) through the use of solutions to auxiliary maximization problems. Obviously this becomes significant in the absolutely-continuous case, where the optimal terminal wealth $\hat{X}_T$ is infinite with strictly positive probability and it no longer has integral representation. What we first do is to define a sequence of problems suitably stopping the original one. This lead us to a sequence of stochastic integrals whose terminal values, equal to the solutions of these new problems, are convergent to $\hat{X}_T$ in probability.

The paper is organized as follows. In Section 2 we briefly recall some basic results in the setting in which our analysis enters. We also introduce a sequence of optimization problems obtained from our original problem (1), and we formulate the main Theorem 2.8. In Section 3 we give a class of security market models, which fit to the absolutely-continuous case. Section 4 contains the proofs of our principal results.

2 The formulation of the main results

Here we sketch out the background to our analysis. First it is convenient to state some results we rely heavily on.

2.1 The original problem

Let us set $I = (U')^{-1} = -V'$, so that $V(y) = U(I(y)) - yI(y)$.

Theorem 2.1 *[9, Theorem 2.2] Let the locally-bounded semimartingale $S = (S_t)_{0\leq t\leq T}$ and the utility function $U : \mathbb{R} \to \mathbb{R}$ satisfy Assumptions 1.1-1.3. Then we have:*

(i) The value functions u(x) and v(y) are conjugate; they are continuously differentiable, strictly concave (resp. convex) on $\mathbb{R}$ (resp. $\mathbb{R}_+$) and satisfy

$$u'(-\infty) = -v'(0) = v'(+\infty) = +\infty, \quad u'(+\infty) = 0. \tag{6}$$

(ii) The optimizers $\hat{X}_T(x)$ and $\hat{\boldsymbol{Q}}_y$ of the problem (1) (resp. (4)) exist, are unique and satisfy

$$\hat{X}_T(x) = I\Big(y\frac{d\hat{\boldsymbol{Q}}_y}{d\boldsymbol{P}}\Big) \text{ or, equivalently, } \frac{d\hat{\boldsymbol{Q}}_y}{d\boldsymbol{P}} = \frac{U'(\hat{X}_T(x))}{y}, \tag{7}$$

where $x \in \mathbb{R}$ and $y \in \mathbb{R}_+$ are related via $y = u'(x)$.

(iii) The following relations hold true:

$$xu'(x) = \boldsymbol{E}[\hat{X}_T(x)U'(\hat{X}_T(x))], v'(y) = \boldsymbol{E}\Big[\frac{d\hat{\boldsymbol{Q}}_y}{d\boldsymbol{P}}V'\Big(y\frac{d\hat{\boldsymbol{Q}}_y}{d\boldsymbol{P}}\Big)\Big]. \tag{8}$$

(iv) If $\hat{\boldsymbol{Q}}_y \in \mathcal{M}^e(S)$ and $x = -v'(y)$, then $\hat{X}_T(x)$ equals the terminal value of a process of the form $\hat{X}_t(x) = x + (\hat{H}(x) \cdot S)_t$, where $\hat{H}$ is predictable and S-integrable, and such that $\hat{X}(x)$ is a uniformly integrable martingale under $\hat{\boldsymbol{Q}}_y$.

From these formulae we also obtain the following one:

$$u(x) = \mathbf{E}[U(\hat{X}_T(x))] = xy + \mathbf{E}\Big[V\Big(y\frac{d\hat{\mathbf{Q}}_y}{d\mathbf{P}}\Big)\Big], \ \ y = u'(x), \tag{9}$$

that we use to formulate the primal problem (1) in a different way. In this order, we also introduce a suitable version of a proposition proved by Biagini and Frittelli[2] in a more general context:

Proposition 2.2 *Let U satisfy Assumption 1.2 and $\boldsymbol{Q}$ be any measure in $\mathcal{M}^a_f(S) = \{\boldsymbol{Q} : \boldsymbol{Q} \in \mathcal{M}^a(S)$ and $\boldsymbol{E}\big[V\big(\frac{d\boldsymbol{Q}}{d\boldsymbol{P}}\big)\big] < \infty\}$. Then, for $x \in \mathbb{R}$, the optimal solution to*

$$\min_{\lambda>0} \lambda x + \boldsymbol{E}\Big[V\Big(\lambda\frac{d\boldsymbol{Q}}{d\boldsymbol{P}}\Big)\Big] \tag{10}$$

is the unique solution to the first order condition

$$x + \boldsymbol{E}\Big[\frac{d\boldsymbol{Q}}{d\boldsymbol{P}}V'\Big(\lambda\frac{d\boldsymbol{Q}}{d\boldsymbol{P}}\Big)\Big] = 0. \tag{11}$$

Noting that relation (9) implies that $y = u'(x)$ solves (11) for $\mathbf{Q} = \hat{\mathbf{Q}}_y$, we get $u'(x)$ as the optimizer to the problem

$$\min_{\lambda>0} \lambda x + \mathbf{E}\Big[V\Big(\lambda\frac{d\hat{\mathbf{Q}}_y}{d\mathbf{P}}\Big)\Big]. \tag{12}$$

This allows us to rewrite the maximization problem (1) as

$$u(x) = \min_{\lambda>0, \mathbf{Q}\in\mathcal{M}^a_f(S)} \lambda x + \mathbf{E}\Big[V\Big(\lambda\frac{d\mathbf{Q}}{d\mathbf{P}}\Big)\Big], \tag{13}$$

which admits unique solution $(\lambda, \mathbf{Q}) = (y, \hat{\mathbf{Q}}_y)$, where, as usual, $y = u'(x)$. In Section 4 it will turn out to be convenient to consider this formulation of the primal problem (1) which, in fact, also involves the dual optimizer.

Let us now take a look at the integral representation in Theorem 2.1-*(iv)*. As mentioned earlier, this characterization of the optimal wealth is submitted to the equivalence of $\hat{\mathbf{Q}}$ to $\mathbf{P}$, whereas the non-equivalent (i.e. the absolutely-continuous) case is left open. In this paper we especially take care of this last situation, that is the one in which our results become significant. Therefore we often assume, or emphasize, the case where the minimax martingale measure is not equivalent to the historical probability. Let us now mention the following known situations in which this fact cannot occur. If $V(0) = U(\infty) = \infty$, formulation (4) of the dual problem makes the minimizer

$\hat{\mathbf{Q}}$ satisfy $\frac{d\hat{\mathbf{Q}}}{d\mathbf{P}} > 0$ $\mathbf{P}$-almost surely, i.e. the optimal measure lies in the set $\mathcal{M}^e(S)$. Furthermore, when there exists an equivalent martingale measure with finite generalized entropy, i.e. when $\mathcal{M}^e_f(S) = \{\mathbf{Q} : \mathbf{Q} \in \mathcal{M}^e(S) \text{ and } \mathbf{E}\big[V\big(\frac{d\mathbf{Q}}{d\mathbf{P}}\big)\big] < \infty\} \neq \emptyset$, under the Inada conditions we obtain again $\hat{\mathbf{Q}}$ equivalent to $\mathbf{P}$ (see Csiszar[3] for the exponential utility and Kabanov-Stricker[6] for the general case).

We now impose a further requirement on our market model, bearing in mind that this condition is satisfied, for example, by the brownian filtration. The general case is to be left for future research.

Assumption 2.3 *The filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfies the usual conditions of saturatedness and right continuity. Moreover, every stopping time is $(\mathcal{F}_t)$-predictable.*

2.2 The auxiliary problems

Recall that our analysis focuses on the absolutely-continuous case. As remarked in the introduction, we aim at providing a sequence of optimization problems which shall permit the approximation of the solution to the original one.

Denoting by $(Z_t)_{0\leq t\leq T}$ the density process corresponding to the optimal martingale measure:

$$Z_0 \equiv 1, \;\; Z_t = \mathbf{E}\Big[\frac{d\hat{\mathbf{Q}}}{d\mathbf{P}}\Big|\mathcal{F}_t\Big], \, t \in (0,T), \;\; Z_T = Z = \frac{d\hat{\mathbf{Q}}}{d\mathbf{P}}, \tag{14}$$

we can define the following stopping times:

$$\tau = \inf\{t > 0 : Z_t = 0\}, \, \tau_n = \inf\{t > 0 : Z_t \leq n^{-1}\}, \, n \in \mathbb{N}, \tag{15}$$

where we put $\inf\emptyset = +\infty$ and $Z_\infty = Z_\tau = Z_T$.

By Assumption 2.3, τ is a predictable time and then there exists a sequence $(\sigma_n)_{n\geq 1}$ of stopping times announcing it: σ_n increasing, $\sigma_n < \tau$, $\forall n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} \sigma_n = \tau$. We can now show that, in fact, τ is announced by $(\tau_n)_{n\geq 1}$ on $\{\tau < \infty\}$.

Proposition 2.4 *Let us define the stopping times*

$$\overline{\tau}_n = \tau_n 1_{\{\tau_n<\infty\}} + n 1_{\{\tau_n=\infty\}}. \tag{16}$$

Under Assumption 2.3, the sequence $(\overline{\tau}_n)_n$ announces τ.

Proof: By (15) and (16) we have $\overline{\tau}_n$ increasing and $\overline{\tau}_n \leq \tau_n \leq \tau$. Moreover $\overline{\tau}_n < \tau$ clearly holds on $\{\tau = \infty\}$ (which equals Ω $\mathbf{P}$-a.s. if we are in the equivalent case). We now consider the absolutely-continuous case and prove that the strict inequality is also true in the set $\{\tau < \infty\}$. If not, there exists $B \subset \{\tau < \infty\}$ with $\mathbf{P}(B) > 0$ and $\tau_n = \overline{\tau}_n = \tau$ on B. Since $(Z_t)_t$ is a uniformly integrable martingale, the optional-sampling theorem and the martingale convergence theorem give us $Z_{\tau-} = \lim_n Z_{\sigma_n} =$

$\lim_n \mathbf{E}[Z_\tau|\mathcal{F}_{\sigma_n}] = \mathbf{E}[Z_\tau|\vee_n\mathcal{F}_{\sigma_n}] = \mathbf{E}[Z_\tau|\mathcal{F}_{\tau-}] = 0$ on $\{\tau < \infty\}$, τ being $\mathcal{F}_{\tau-}$-measurable. On the other hand, since $\overline{\tau}_n > 0$, $Z_{\overline{\tau}_n-} \geq n^{-1}$ by definition. We then obtain $0 = Z_{\tau-}1_B = Z_{\overline{\tau}_n-}1_B \geq n^{-1}1_B$ and this contradiction proves $\overline{\tau}_n < \tau$ a.s. on Ω. In order to end the proof, there still remains to show that $\overline{\tau}_n$ (or, equivalently, τ_n) converges to τ. By monotonicity $\eta = \lim_n \tau_n \leq \tau$ is well defined and, of course, $\eta \geq \tau_n$. We now show that this limit, in fact, equals τ. Since on $\{\eta = \infty\}$ this is clearly true, we consider the set $\{\eta < \infty\}$ (where, $\forall n \in \mathbb{N}$, $\tau_n < \infty$ too). The optional-sampling theorem gives us $\mathbf{E}[Z_\eta 1_{\{\tau_n<\infty\}}] = \mathbf{E}[Z_{\tau_n}1_{\{\tau_n<\infty\}}] \leq n^{-1}$, since on $\{\tau_n = \infty\}$ we have $\eta = \tau_n = \infty$ and $Z_\eta = Z_{\tau_n} = Z_T$. Hence, by applying Chebyshev's inequality, $\mathbf{P}(\{Z_\eta 1_{\{\tau_n<\infty\}} \geq c\}) \leq (cn)^{-1}$ for any constant $c > 0$ we fix. This yields $\lim_n \mathbf{P}(\{Z_\eta 1_{\{\tau_n<\infty\}} \geq c\}) = 0$, $\forall c > 0$, i.e. $Z_\eta 1_{\{\tau_n<\infty\}}$ tends to 0 in probability. It follows from the dominated convergence that $\| Z_\eta 1_{\{\tau_n<\infty\}} \|_{L^1(\mathbf{P})} \to 0$ and then we get $Z_\eta 1_{\{\eta<\infty\}} = 0$, since $\mathbf{E}[Z_\eta 1_{\{\eta<\infty\}}] \leq \mathbf{E}[Z_\eta 1_{\{\tau_n<\infty\}}]$. This fact leads us to conclude that $\eta = \tau$ by (15), making our proof complete. □

This proposition clearly states the continuity of the density process $(Z_t)_t$ at τ. Indeed, the right continuity of the filtration yields the right continuity of any uniformly integrable martingale process, and the assertion of the proposition gives us $\lim_{t\uparrow\tau} Z_t = \lim_{n\to\infty} Z_{\tau_n} = Z_\tau$.

Remark 2.5 As pointed out by the referee, in the proof of Proposition 2.4 we actually don't use the specific fact that $(Z_t)_t$ is the density process of the dual minimizer, which allows us to reformulate the proposition in a more general way. By doing so, we obtain the following result, interesting by itself.

Proposition 2.6 *Let $M = (M_t)_{0\leq t\leq T}$ be a non-negative uniformly integrable martingale in a filtered probability space $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, \boldsymbol{P})$, with $T \in (0, \infty]$ and $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the usual conditions. Let ρ be the first time in which M reaches zero and assume that it is a predictable stopping time. Then ρ is announced by the sequence of stopping times*

$$\overline{\rho}_n = \rho_n 1_{\{\rho_n<\infty\}} + n 1_{\{\rho_n=\infty\}},\ n \in \mathbb{N}, \tag{17}$$

where

$$\rho_n = \inf\{t > 0 : M_t \leq n^{-1}\}. \tag{18}$$

Remark 2.7 In addition to what we have concluded above, it is not unworthy to underline that $\hat{X}_T(x), Z_T \in L^0(\Omega, \mathcal{F}_{\tau-}, \mathbf{P})$ and relation

$$u_{\tau-}(x) \triangleq \sup_H \mathbf{E}[U(x + (H\cdot S)_{\tau-})] = u(x) \tag{19}$$

holds true. This means that we can visualize the optimization problem (1) in $[0,T]$, as it was defined in the random interval $[0,\tau[$. Of course, the optimizers of the relative dual problems coincide, too:

$$v_{\tau-}(y) \triangleq \inf_{\mathcal{M}^a(S^{\tau-})} \mathbf{E}\Big[V\Big(y\frac{d\mathbf{Q}}{d\mathbf{P}}\Big)\Big] = v(y), \quad \text{where} \quad y = u'(x) = u'_{\tau-}(x)$$

and $\mathcal{M}^a(S^{\tau-})$ refers to the stopped process $(S_t^{\tau-})_{0\le t\le T} = (S_{t\wedge\tau-})_{0\le t\le T}$. Therefore, from now on, we regard problems (1) and (19) as indistinguishable.

Let us now consider the trading random interval $[0,\tau_n]$, for any n in $\mathbb{N}$. We define the expected utility maximization problem relative to it:

$$u_n(x) = \sup_H \mathbf{E}[U(x + (H\cdot S)_{\tau_n})], \quad x\in\mathbb{R} \tag{20}$$

as well as the dual one:

$$v_n(y) = \inf_{\mathbf{Q}\in\mathcal{M}^a(S^{\tau_n})} \mathbf{E}\Big[V\Big(y\frac{d\mathbf{Q}}{d\mathbf{P}}\Big)\Big], \quad y\in\mathbb{R}_+ \tag{21}$$

where $\mathcal{M}^a(S^{\tau_n})$ refers to the stopped process $(S_t^{\tau_n})_{0\le t\le T} = (S_{t\wedge\tau_n})_{0\le t\le T}$. Of course $(u_n)_n$ is increasing and $u_n \le u$. Note that (20) can be reformulated in the following equivalent ways:

$$u_n(x) = \sup_H \mathbf{E}[U(x + (H\cdot S^{\tau_n})_T)] = \sup_H \mathbf{E}[U(x + (H1_{[0,\tau_n]}\cdot S)_T)].$$

Throughout the paper we will use the notation $X_T^{(n)}(x) = X_{\tau_n}^{(n)}(x)$ and $\mathbf{Q}_y^{(n)}$ for the optimal terminal wealth of the primal problem (20) and for the martingale measure solving (21), respectively (the dependence on x or y being dropped when it does not generate confusion). It visibly follows that the optimal solutions relative to the problems in $[0,\tau_n]$, satisfy relations analogous to the ones in (7):

$$X_T^{(n)}(x) = I\Big(y_n\frac{d\mathbf{Q}_{y_n}^{(n)}}{d\mathbf{P}}\Big) \quad \text{and} \quad \frac{d\mathbf{Q}_{y_n}^{(n)}}{d\mathbf{P}} = \frac{U'(X_T^{(n)}(x))}{y_n}, \tag{22}$$

where $y_n = u_n'(x) = \mathbf{E}[U'(X_T^{(n)}(x))]$.

The main theorem of the paper can now be stated as follows.

Theorem 2.8 *Assume $U:\mathbb{R}\to\mathbb{R}$ and $(S_t)_{0\le t\le T}$ to satisfy Assumptions 1.1-1.3, 2.3. Then the following relations between the solutions to the original problems and the auxiliary ones hold true:*

(i) $u_n(x) \underset{n}{\longrightarrow} u(x)$*;*

(ii) $X_T^{(n)}(x) \xrightarrow[n]{\boldsymbol{P}} \hat{X}_T(x)$, *and* $\frac{d\boldsymbol{Q}_{y_n}^{(n)}}{d\boldsymbol{P}} \xrightarrow[n]{L^1(\boldsymbol{P})} \frac{d\hat{\boldsymbol{Q}}_y}{d\boldsymbol{P}}$.

The theorem states that the optimal wealth $\hat{X}_T$ (solution to the maximization problem in $[0, T]$) can be approximated through the optimal wealths $X_T^{(n)}$ (reachable by trading up to random times τ_n only). Clearly the interesting case is the absolutely-continuous one. Indeed, in this case $\hat{X}_T$ admits integral representation only $\hat{\mathbf{Q}}$-almost surely (see Theorem 2.1-*(iv)*), but, considering Theorem 2.8 in conjunction with Lemma 4.1, $\hat{X}_T$ is achievable as the limit of terminal values for some stochastic integrals.

3 A class of examples

In this section we construct a class of examples showing how, for any utility function satisfying our requests, the absolutely-continuous case may occur.

Let $U : \mathbb{R} \to \mathbb{R}$ be an utility function satisfying Assumption 1.2. We construct a real-valued (locally) bounded semimartingale $S = (S_n)_{n \in \mathbb{N}_0}$, based on and adapted to a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in \mathbb{N}_0}, \mathbf{P})$ describing the discounted price process of a risky traded asset. Assumption 2.3, in this setting, is clearly satisfied. We shall give conditions on $\mathbf{P}$ and U, so that the financial market modelled by S satisfies Assumptions 1.1, 1.3 and fits to the absolutely-continuous case. This means that, for an economic agent investing in this market, the optimization problem (4) returns a martingale measure lying in $\mathcal{M}^a(S)$ but not in $\mathcal{M}^e(S)$. Moreover, the optimal terminal wealth will have the nice representation $(\hat{H} \cdot S)_\infty = (1 \cdot S)_\infty = \lim_{n \to \infty} S_n$ where this limit exists, whereas will be equal to $+\infty$ otherwise.

3.1 Construction of the asset price process

To simplify the notation we fix the initial endowment at $x = 0$, so that the primal problem takes the form

$$u(0) = \sup_H \mathbf{E}[U((H \cdot S)_\infty)]. \tag{23}$$

In order to define the asset price process S, we choose a trinomial tree model in which, at every step n, one can "go up", "go down" or remain at the same level (see [10] for a similar schema). To this end, we consider two sequences $(a_n)_{n=0}^\infty$ and $(b_n)_{n=1}^\infty$ of real numbers such that $a_1 = 0$, sgn(a_n)=sgn(b_n)=sgn(n) $\forall n \in \mathbb{N}$, $|b_n| > |a_n|$ and $|a_n|$ (so $|b_n|$ too) increases to $+\infty$ as $n \to +\infty$. Let us put $S_0 = 0$ in $\Omega =: C_0$, then split C_0 in tree sets, say A_1 (where S remains at the same level 0), B_1 (where the process goes down at b_1) and C_1 (where it goes up at a_2). In the same way, for any n in $\mathbb{N}$, we put $S_n = S_{n-1}$ in $\bigcup_{k=1}^{n-1}(A_k \cup B_k)$, whereas C_{n-1} is split into three sets, say A_n, B_n and C_n, where we define S_n equal to a_n, b_n and a_{n+1}, respectively. Therefore, at time n with n even (resp. odd), the process can go up, if B_n (resp. C_n) occurs, it can go down, if C_n (resp. B_n) occurs, or remain at the same level, in the sets $A_1, ..., A_n, B_1, ..., B_{n-1}$. In this way the process we have constructed works as follows, for any $n \in \mathbb{N}$:

$$S_n = \begin{cases} a_k, & \text{on } A_k,\ k = 1, .., n, \\ b_k, & \text{on } B_k,\ k = 1, .., n, \\ a_{n+1}, & \text{on } C_n. \end{cases}$$

Moreover, as n goes to $+\infty$, S admits path-wise limit on $\bigcup_{n=1}^{\infty}(A_n \cup B_n)$, whereas it oscillates between $+\infty$ and $-\infty$ on $C_\infty = \bigcap_{n=0}^{\infty} C_n$. Since we will put C_∞ not null under the historical probability, the limit of the process is not almost surely well defined and it becomes convenient to introduce the random variable

$$S_\infty = \lim_n S_n 1_{C_\infty^c}.$$

This construction follows an analogue pattern such as the one in [10] and here we can give a similar interpretation of the asset price process. Indeed, we may consider S as the value of a player suitably-stopped portfolio, when the game consists of a sequence of independent experiments with three outcomes, say

$$\begin{cases} u_n : & \text{to go up in the } n^{th} \text{ trial,} \\ m_n : & \text{to remain at the same level in the } n^{th} \text{ trial,} \\ d_n : & \text{to go down in the } n^{th} \text{ trial.} \end{cases}$$

The value of the game at $n = 0$ is fixed equal to zero, and the increments we consider are:

$$\eta_n = \begin{cases} a_{n+1} - a_n, & \text{if } u_n \text{ and } n \text{ odd, or } d_n \text{ and } n \text{ even, occurs,} \\ 0, & \text{if } m_n \text{ occurs,} \\ b_n - a_n, & \text{if } d_n \text{ and } n \text{ odd, or } u_n \text{ and } n \text{ even, occurs.} \end{cases}$$

Hence, as long as the player continues the game, his portfolio value at time $n \in \mathbb{N}$ is

$$M_n = \sum_{k=1}^{n} \eta_k.$$

Now the idea is to play as long as experiments have outcomes of up-type when n odd and of down-type when n even, while the game stops at the first time in which it doesn't occur. Let us introduce the stopping times

$$\nu := \inf\{n : m_n \text{ occurs}\}, \sigma := \inf\{n : d_n \text{ and } n \text{ odd, or } u_n \text{ and } n \text{ even, occurs}\}, \rho := \nu \wedge \sigma.$$

If we define our sets as follows:

$$A_n = \{n = \nu < \sigma\},\ B_n = \{n = \sigma < \nu\},\ C_n = \{\rho > n\} \text{ and } \quad C_\infty = \{\rho = \infty\}$$

and allow the gambler to play up to the random time ρ, his portfolio value turns out to be modelled by the stopped process

$$M_n^\rho = S_n.$$

After this comment, let us come back to the definition of our model and denote by $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ the natural filtration generated by S:
$\mathcal{F}_0 = \{\emptyset, \Omega\}$ is the trivial algebra,
$\mathcal{F}_n = \sigma(S_n) = \sigma(\{A_1, .., A_n, B_1, .., B_n, C_n\}), \forall n \in \mathbb{N}$ and
$\mathcal{F} = \mathcal{F}_\infty = \bigvee_n \mathcal{F}_n = \sigma(\{(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}}, C_\infty\})$.

3.2 Characterization of the martingale measures

The unique condition needed on the sequences $(a_n)_n$ and $(b_n)_n$ will be obtained from the characterization of the martingale measures for the process S (and independently of the utility function U we consider). Then, before we introduce the assumption on the historical probability $\mathbf{P}$, we proceed to identify the set $\mathcal{M}^a(S)$. Of course every measure $\mathbf{Q} \in \mathcal{M}^a(S)$ has to satisfy the martingale property

$$\mathbf{Q}(B_n)(b_n - a_n) + \mathbf{Q}(C_n)(a_{n+1} - a_n) = 0, \quad \forall n \in \mathbb{N}, \tag{24}$$

that is

$$\frac{\mathbf{Q}(B_n)}{\mathbf{Q}(C_n)} = \frac{a_n - a_{n+1}}{b_n - a_n} =: \xi_n, \quad \forall n \in \mathbb{N}. \tag{25}$$

To have $\mathcal{M}^e(S) \neq \emptyset$, we need a probability measure $\mathbf{Q}$ satisfying (25) and such that $\mathbf{Q} \sim \mathbf{P}$. This measure clearly satisfies $\lim_n \mathbf{Q}(B_n) = 0$ and, requiring $\mathbf{P}(C_\infty) > 0$, also $\lim_n \mathbf{Q}(C_n) > 0$, so that $\xi_n \underset{n}{\to} 0$. More precisely, at every step n we can rewrite: $\mathbf{Q}(B_n) = \xi_n \mathbf{Q}(C_n)$ and $\mathbf{Q}(A_n) = \gamma_n \mathbf{Q}(C_n)$ for some γ_n, so that $\mathbf{Q}(C_{n-1}) = (1 + \gamma_n + \xi_n)\mathbf{Q}(C_n)$. Now, since we want

$$0 < \mathbf{Q}(C_\infty) = \lim_n \mathbf{Q}(C_n) = \lim_n \prod_{k=1}^{n} \frac{1}{1 + \gamma_k + \xi_k} = \prod_{n=1}^{\infty} \frac{1}{1 + \gamma_n + \xi_n},$$

we need $\sum_{n=1}^\infty \left(1 - \frac{1}{1+\gamma_n+\xi_n}\right) < \infty$ or, equivalently, $\frac{\gamma_n+\xi_n}{1+\gamma_n+\xi_n} \downdownarrows 0$. Here, given two sequences $(f_n)_n \subset \mathbb{R}$ and $(g_n)_n \subset \mathbb{R} \setminus \{0\}$, by the notation $\frac{f_n}{g_n} \downdownarrows 0$ or $f_n \ll g_n$ we mean that $\sum_n \frac{f_n}{g_n} < \infty$, whereas by $f_n \approx g_n$ we indicate that $\frac{f_n}{g_n} \in [c^{-1}, c]$, asymptotically, for some $c > 1$. Arranging things such that $\mathbf{Q}(A_n)$ tends to zero sufficiently quickly ($\gamma_n \ll \xi_n$), we can relax the last requirement to the following

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{1 + \xi_n}\right) = \sum_{n=1}^{\infty} \frac{a_n - a_{n+1}}{b_n - a_{n+1}} < \infty. \tag{26}$$

We then obtain that the corresponding martingale measure $\mathbf{Q}$ lies in $\mathcal{M}^e(S)$.

To have (26) satisfied it's sufficient to have

$$|a_{n+1}| \ll |b_n|, \tag{27}$$

so we may and do assume it holds true. In this way, Assumption 1.1 turns out to be satisfied. Requirement (27) is the only one we make on the sequences $(a_n)_n$ and $(b_n)_n$, and we point out that it does not depend on the utility function U describing the agent preferences. To give an example, a good choice for these parameters is $a_n \approx \mathrm{sgn}(n)n$ and $b_n \approx \mathrm{sgn}(n)2^n$.

3.3 The historical probability

We now put some conditions on the historical probability **P**, to make the solution of (23) having the announced representation. What we want is the optimal wealth to be equal to $(\hat{H} \cdot S)_\infty = (1 \cdot S)_\infty = S_\infty$, on C_∞^c, and to $+\infty$ on C_∞ (where $\mathbf{P}(C_\infty) > 0$). This result motivates the following requirements:

(P1) $\mathbf{P}(A_n)U'(a_n) \downdownarrows 0$ and $\mathbf{P}(B_n)U'(b_n) \downdownarrows 0$;

(P2) $\mathbf{P}(A_{n+1})U'(a_{n+1}) \approx \sum_{k=n+1}^{\infty} \big(\mathbf{P}(A_k)U'(a_k) + \mathbf{P}(B_k)U'(b_k)\big)$;

(P3) $\mathbf{P}(B_n)U'(b_n) = \xi_n \sum_{k=n+1}^{\infty} \big(\mathbf{P}(A_k)U'(a_k) + \mathbf{P}(B_k)U'(b_k)\big)$,
so that, if (P2) holds, $\mathbf{P}(B_n)U'(b_n) \approx \xi_n \mathbf{P}(A_{n+1})U'(a_{n+1})$;

(P4) $|a_n|\mathbf{P}(A_n)U'(a_n) \downdownarrows 0$,
which, if (P2) and (P3) hold, implies $|b_n|\mathbf{P}(B_n)U'(b_n) \downdownarrows 0$, by (27);

(P5) $\mathbf{P}(A_n)U(-|a_n|) \downdownarrows 0$ and $\mathbf{P}(B_n)U(-|b_n|) \downdownarrows 0$;

(P6) $\sum_{n=1}^{\infty} \big(\mathbf{P}(A_n) + \mathbf{P}(B_n)\big) < 1$ (i.e. $\mathbf{P}(C_\infty) > 0$, as mentioned earlier).

Note that, since $|a_n|$ and $|b_n|$ increase to $+\infty$, (P4) is clearly stronger than (P1) and then the requirements we make on **P** are (P2)-(P6). To give a concrete example, consider the preferences of the agent as modelled by the exponential utility function $U(x) = -e^{-x}$ and, as before, put $a_n \approx \mathrm{sgn}(n)n$ and $b_n \approx \mathrm{sgn}(n)2^n$. In this setting, a good choice for the measure **P** is $\mathbf{P}(A_n) \approx e^{-2^n + a_n}$ and $\mathbf{P}(B_n) \approx n2^{-n}e^{-2^{n+1}+b_n}$, so that (P2)-(P6) are satisfied.

3.4 The minimax martingale measure

Let us define a measure $\bar{\mathbf{Q}}$, absolutely continuous w. r. to **P**, in the following way:

$$\frac{d\bar{\mathbf{Q}}}{d\mathbf{P}} = \frac{U'(X_\infty)}{c}, \quad \text{where} \quad X_\infty = \begin{cases} S_\infty, & \text{on } C_\infty^c \\ +\infty, & \text{on } C_\infty \end{cases} \tag{28}$$

and $c = ||U'(X_\infty)||_{L^1(\mathbf{P})} < \infty$ by (P1). Therefore we have $\bar{\mathbf{Q}}(A_n) = c^{-1}U'(a_n)\mathbf{P}(A_n)$, $\bar{\mathbf{Q}}(B_n) = c^{-1}U'(b_n)\mathbf{P}(B_n)$, $\bar{\mathbf{Q}}(C_\infty) = c^{-1}U'(\infty)\mathbf{P}(C_\infty) = 0$ and, in particular, $\bar{\mathbf{Q}}$ is not equivalent to $\mathbf{P}$. Moreover, by (P3),

$$\bar{\mathbf{Q}}(B_n) = \xi_n \sum_{k=n+1}^{\infty} \big(\bar{\mathbf{Q}}(A_k) + \bar{\mathbf{Q}}(B_k)\big) = \xi_n \bar{\mathbf{Q}}(C_n), \tag{29}$$

i.e. $\bar{\mathbf{Q}}$ satisfies the martingale property (25). It follows that $\bar{\mathbf{Q}}$ is a good candidate to be the minimax martingale measure $\hat{\mathbf{Q}}_c$ and we will show that, under our assumptions, this will be the case. Before proving the optimality of $\bar{\mathbf{Q}}$, we show that the following properties hold true:

- $X_\infty \in L^1(\bar{\mathbf{Q}})$ with $\mathbf{E}_{\bar{\mathbf{Q}}}[X_\infty] = 0$;
- $X_\infty \in L^1(\mathbf{Q})$ with $\mathbf{E}_{\mathbf{Q}}[X_\infty] = 0, \quad \forall \mathbf{Q} \in \mathcal{M}^a_f(S)$.

By (28) and (P4) we clearly have $\mathbf{E}_{\bar{\mathbf{Q}}}[|X_\infty|] = \mathbf{E}_{\bar{\mathbf{Q}}}[|S_\infty|] = \sum_{n=1}^{\infty} \big(|a_n|\bar{\mathbf{Q}}(A_n) + |b_n|\bar{\mathbf{Q}}(B_n)\big) < \infty$ and also $\mathbf{E}_{\bar{\mathbf{Q}}}[|X_\infty - S_n|] \underset{n\to\infty}{\longrightarrow} 0$, i.e. $S_n \overset{L^1(\bar{\mathbf{Q}})}{\longrightarrow} X_\infty$, by approximation in (P2). This yields $(S_n)_n$ uniformly integrable w.r. to $\bar{\mathbf{Q}}$, with

$$\mathbf{E}_{\bar{\mathbf{Q}}}[X_\infty|\mathcal{F}_n] = \mathbf{E}_{\bar{\mathbf{Q}}}[S_\infty|\mathcal{F}_n] = S_n \text{ and } \mathbf{E}_{\bar{\mathbf{Q}}}[X_\infty] = \mathbf{E}_{\bar{\mathbf{Q}}}[S_\infty] = 0, \tag{30}$$

as claimed. Let now $\mathbf{Q}$ be any measure in $\mathcal{M}^a_f(S)$. Since
$\mathbf{E}\big[V\big(\frac{d\mathbf{Q}}{d\mathbf{P}}\big)\big] = \sum_{n=1}^\infty \big(V\big(\frac{\mathbf{Q}(A_n)}{\mathbf{P}(A_n)}\big)\mathbf{P}(A_n) + V\big(\frac{\mathbf{Q}(B_n)}{\mathbf{P}(B_n)}\big)\mathbf{P}(B_n)\big) + V\big(\frac{\mathbf{Q}(C_\infty)}{\mathbf{P}(C_\infty)}\big)\mathbf{P}(C_\infty)$
is a finite quantity and the function V is bounded from below, we have $V\big(\frac{\mathbf{Q}(A_n)}{\mathbf{P}(A_n)}\big)\mathbf{P}(A_n) \downdownarrows 0$ and $V\big(\frac{\mathbf{Q}(B_n)}{\mathbf{P}(B_n)}\big)\mathbf{P}(B_n) \downdownarrows 0$. Therefore, using (3) for $y = \dfrac{d\mathbf{Q}}{d\mathbf{P}}$ and $x_n = -|a_n|$ or $-|b_n|$, we obtain

$$|a_n|\mathbf{Q}(A_n) \le V\Big(\frac{\mathbf{Q}(A_n)}{\mathbf{P}(A_n)}\Big)\mathbf{P}(A_n) - U(-|a_n|)\mathbf{P}(A_n) \downdownarrows 0 \quad \text{and}$$

$$|b_n|\mathbf{Q}(B_n) \le V\Big(\frac{\mathbf{Q}(B_n)}{\mathbf{P}(B_n)}\Big)\mathbf{P}(B_n) - U(-|b_n|)\mathbf{P}(B_n) \downdownarrows 0,$$

by assumption (P5). This yields the integrability of X_∞ w.r. to every $\mathbf{Q}$ in $\mathcal{M}^a_f(S)$, with $\mathbf{E}_{\mathbf{Q}}[X_\infty] = \mathbf{E}_{\mathbf{Q}}[S_\infty] = 0$, and in particular $\mathbf{Q}(C_\infty) = 0$ (so that $\mathbf{Q} \ll \bar{\mathbf{Q}}$). Now we are able to prove the optimality of the measure $\bar{\mathbf{Q}}$ in the set $\mathcal{M}^a(S)$. Indeed, for any given probability measure $\mathbf{Q} \in \mathcal{M}^a_f(S)$:

$$\begin{aligned}\mathbf{E}\Big[V\Big(c\frac{d\mathbf{Q}}{d\mathbf{P}}\Big) - V\Big(c\frac{d\bar{\mathbf{Q}}}{d\mathbf{P}}\Big)\Big] &\ge \mathbf{E}\Big[V'\Big(c\frac{d\bar{\mathbf{Q}}}{d\mathbf{P}}\Big)\Big(c\frac{d\mathbf{Q}}{d\mathbf{P}} - c\frac{d\bar{\mathbf{Q}}}{d\mathbf{P}}\Big)\Big] \\ &= c\mathbf{E}_{\mathbf{Q}}\Big[-(U')^{-1}\Big(c\frac{d\bar{\mathbf{Q}}}{d\mathbf{P}}\Big)\Big] - c\mathbf{E}_{\bar{\mathbf{Q}}}\Big[-(U')^{-1}\Big(c\frac{d\bar{\mathbf{Q}}}{d\mathbf{P}}\Big)\Big] \\ &= c\mathbf{E}_{\mathbf{Q}}[-X_\infty] - c\mathbf{E}_{\bar{\mathbf{Q}}}[-X_\infty] = 0,\end{aligned}$$

by the convexity of V. Consequently $\bar{\mathbf{Q}}$ turns out to be the optimal solution $\hat{\mathbf{Q}}_c$ to the dual problem. On the other hand, by (8) we have $u'(0) = (-v')^{-1}(0) = c$. Then the $\bar{\mathbf{Q}} \in \mathcal{M}^a(S)$ we have defined is the optimal martingale measure $\hat{\mathbf{Q}}_{u'(0)}$ and its density is proportional to the marginal utility of X_∞. Moreover, by (28) and (P6), this martingale measure is not equivalent to the probability measure $\mathbf{P}$. By this fact, and guided by relation (7), we obtain that the optimal terminal wealth $\hat{X}_\infty(0)$ equals X_∞, which is infinite with strictly positive probability by (P6). Then (23) produces $u(0) = \mathbf{E}[U(X_\infty)] < \infty$ by assumption (P5).

3.5 On the approximation of the optimal wealth

We can observe that a sequence $(K^m)_{m\in\mathbb{N}}$ of admissible trading strategies permitting the approximation of the optimal wealth, is given by $K^m = 1_{[0,2m-1]}$. In this way, we obtain processes $Y^m := (K^m \cdot S)$ such that $Y^m_\infty = (K^m \cdot S)_{2m-1} = S_{2m-1} \underset{m}{\rightarrow} \hat{X}_\infty(0)$ and also $\mathbf{E}[U(Y^m_\infty)] \underset{m}{\rightarrow} \mathbf{E}[U(\hat{X}_\infty(0))] = u(0)$. On the contrary, we can easily see that $\hat{X}_n(0) := \mathbf{E}_{\hat{\mathbf{Q}}}[\hat{X}_\infty(0)|\mathcal{F}_n] = S_n \underset{n}{\nrightarrow} \hat{X}_\infty(0)$. Also regarding the auxiliary optimization problems introduced in Section 2, we get $X^{(n)}_\infty(0) = (H^n \cdot S)_{\tau_n} \underset{n}{\rightarrow} \hat{X}_\infty(0)$, by our main theorem, but $\hat{X}_{\tau_n}(0) := \mathbf{E}_{\hat{\mathbf{Q}}}[\hat{X}_\infty(0)|\mathcal{F}_{\tau_n}] \underset{n}{\nrightarrow} \hat{X}_\infty(0)$. Let us now consider the initial interpretation of S as a player portfolio value, in order to reinterpret the processes Y^m now defined. For any n in $\mathbb{N}$, we have $Y^m_n = S_{n\wedge(2m-1)} = M^{\rho\wedge\rho_m}_n$, where ρ_m is the deterministic stopping time $\rho_m = 2m-1$. Hence, we can approximate the optimal terminal wealth through portfolio values of players which stop at time $\rho\wedge\rho_m$ the game described at the beginning. In other words, playing this game (i.e. trading in our market S) we can obtain a wealth that, with strictly positive probability ($> \mathbf{P}(C_\infty)$), is as large as we want.

Remark 3.1 About the choice of the time horizon $+\infty$, we point out that it is not relevant. To have the same examples in a finite trading horizon T, it is sufficient to consider the time scale $T\frac{n}{n+1}$ instead of n.

Remark 3.2 Regarding the unboundedness of the asset price process, we can see how to drop it, leaving unchanged the optimal wealth process $\hat{X}(0)$. Let us define a new process $S^b = (S^b_n)_{n\in\mathbb{N}}$ as follows:

$$S^b_0 = 0 \quad \text{and} \quad S^b_n - S^b_{n-1} = \Delta S^b_n = c_n \Delta \hat{X}_n(0) = c_n \Delta S_n.$$

Of course, for arbitrary constants $c_n \neq 0$, the optimal wealth process obtained by trading in the market S^b is the same as the one obtained by trading in S. So we can choose some constants c_n such that S^b is a uniformly bounded process, for example $c_n = \frac{2^{-(n+1)}}{|b_{n+1}|}$, which gives us $|S^b_n| \leq 1 - 2^{-n}$, $\forall n \in \mathbb{N}$, and also $|S^b_\infty| \leq 1$.

4 Approximation of the optimal solutions

Let us recall the notation introduced in Section 2 for the auxiliary problems. With u_n and v_n we respectively denote the value functions of the primal and the dual problems in the trading interval $[0, \tau_n],\ n \in \mathbb{N}$, whereas with $X_T^{(n)}(x)$ and $\mathbf{Q}_{y_n}^{(n)}$ $(y_n = u_n'(x))$ we mean their optimal solutions. As for the original problems, when it does not generate confusion, we omit the dependence of the solutions on x (resp. on $y_n = u_n'(x)$). Before we prove our main theorem, we need some preparatory results, stated and proved in the next subsection.

4.1 Preliminary results

The following is a very simple, basic observation:

Lemma 4.1 *The optimal martingale measure $\mathbf{Q}_{y_n}^{(n)}$ is equivalent to* **P** *for any n in $\mathbb{N}$.*

Proof: As mentioned earlier, under the Inada conditions, if there exists an equivalent martingale measure with finite generalized entropy then the optimal measure is equivalent to the historical probability (see [6]). We can show that, in fact, this is the case for the optimization problem in $[0, \tau_n]$. To this end it is sufficient to prove that the measure $\hat{\mathbf{Q}}_y$ (optimal solution to the dual problem in $[0, T]$) restricted to the σ-algebra $\mathcal{F}_{\tau_n}$: $\frac{d\hat{\mathbf{Q}}_y}{d\mathbf{P}}\big|_{\mathcal{F}_{\tau_n}} = \mathbf{E}\big[\frac{d\hat{\mathbf{Q}}_y}{d\mathbf{P}}\big|\mathcal{F}_{\tau_n}\big]$, belongs to $\mathcal{M}_f^e(S^{\tau_n})$. Indeed, obviously $\frac{d\hat{\mathbf{Q}}_y}{d\mathbf{P}}\big|_{\mathcal{F}_{\tau_n}} > 0$ a.s. and, by Jensen's inequality, $\mathbf{E}\big[V\big(y\frac{d\hat{\mathbf{Q}}_y}{d\mathbf{P}}\big|_{\mathcal{F}_{\tau_n}}\big)\big] = \mathbf{E}\big[V\big(\mathbf{E}\big[y\frac{d\hat{\mathbf{Q}}_y}{d\mathbf{P}}\big|\mathcal{F}_{\tau_n}\big]\big)\big] \leq \mathbf{E}\big[V\big(y\frac{d\hat{\mathbf{Q}}_y}{d\mathbf{P}}\big)\big] < \infty$, as claimed. □

Then, from Theorem 2.1, the optimal solution to the problem (21) can be written in the following way

$$X_{\tau_n}^{(n)}(x) = x + (H^n \cdot S)_{\tau_n}. \tag{31}$$

Here $X_{\tau_n}^{(n)}(x)$ is finite **P**-a.s., $H^n = H^n 1_{[0,\tau_n]}$ is a predictable S^{τ_n}-integrable process and $(H^n \cdot S)_{0 \leq t \leq \tau_n}$ is a $\mathbf{Q}_{y_n}^{(n)}$-u.i. martingale. It is of fundamental importance to consider the statements of Theorem 2.8 and the characterization (31) of the optimal wealths $X^{(n)}$ together. Indeed this makes the optimal wealth $\hat{X}_T$ (solution of the original problem) attainable as the limit of suitable-portfolios terminal values, revealing the goodness of our approximation.

Let us introduce the sequence

$$(y_n Z^{(n)})_{n\in\mathbb{N}} = \Big(y_n \frac{d\mathbf{Q}_{y_n}^{(n)}}{d\mathbf{P}}\Big)_{n\in\mathbb{N}}$$

of positive measures. We want to prove that we can extract a sequence of convex combinations of them, converging to $yZ = u'(x)\frac{d\hat{\mathbf{Q}}_y}{d\mathbf{P}}$ in probability, where, as usual, y

and y_n are the first derivatives of the value functions u and u_n at a fixed point $x \in \mathbb{R}$. This is just a preliminary result, as we shall only be able in the next subsection to show that the sequence $(y_n Z^{(n)})_n$ itself converges to yZ.

Recall that functions $U, u, \{u_n\}_n$ are increasing, concave and finite valued on $\mathbb{R}$, and that the sequence $(u_n)_n$ is increasing too. Moreover, relation $U \leq u_n \leq u$, $\forall n \in \mathbb{N}$, is clearly satisfied. Hence, for any fixed $x \in \mathbb{R}$, $(y_n)_n = (u_n'(x))_n$ is a bounded sequence, say $y_n \leq \xi \in \mathbb{R}\ \forall n \in \mathbb{N}$, where of course $\xi = \xi(x)$. It follows that even $(y_n Z^{(n)})_n$ is a bounded sequence, lying in $L^1_+(\Omega, \mathcal{F}, \mathbf{P})$, and we can use an appropriate version of Komlos' theorem (see [5]). This produces a sequence $(g_n)_{n\in\mathbb{N}}$ of positive measures converging in probability to some $g \in L^1_+(\Omega, \mathcal{F}, \mathbf{P})$. More precisely, we have

$$g_n = \sum_{k=n}^{\infty} \alpha_k^n y_k Z^{(k)} \in \text{ conv}\,(y_n Z^{(n)}, y_{n+1} Z^{(n+1)}, ...),\ \ n \in \mathbb{N}, \tag{32}$$

with $0 \leq \alpha_k^n \leq 1,\ \sum_{k=n}^{\infty} \alpha_k^n = 1,$ and $g_n \xrightarrow{\mathbf{P}} g \in L^1_+(\Omega, \mathcal{F}, \mathbf{P})$.

It is convenient to introduce the probability measures related to these random variables:

$$\frac{d\mathbf{R}^n}{d\mathbf{P}} = \frac{g_n}{\mathbf{E}[g_n]} = \frac{g_n}{\gamma_n}, \quad \frac{d\mathbf{R}}{d\mathbf{P}} = \frac{g}{\mathbf{E}[g]} = \frac{g}{\gamma}, \quad \gamma_n, \gamma \in (0, \infty). \tag{33}$$

As an immediate consequence of the boundedness of $(y_n)_n$, we have that $(\gamma_n)_n$ is bounded too. Indeed Fatou's lemma gives us

$$\gamma_n = \mathbf{E}\Big[\sum_{k=n}^{\infty} \alpha_k^n y_k Z^{(k)}\Big] \leq \sum_{k=n}^{\infty} \mathbf{E}[\alpha_k^n y_k Z^{(k)}] = \sum_{k=n}^{\infty} \alpha_k^n y_k \leq \xi \tag{34}$$

and also

$$\gamma = \mathbf{E}[\lim_n g_n] \leq \lim_n \mathbf{E}[g_n] = \lim_n \gamma_n \leq \xi. \tag{35}$$

Moreover, since the function V is convex and bounded from below ($V \leq U(0)$ by (3)), we obtain

$$\mathbf{E}[V(g_n)] = \mathbf{E}\Big[V\Big(\lim_p \sum_{k=n}^{p} \alpha_k^n y_k Z^{(k)}\Big)\Big] \leq \sum_{k=n}^{\infty} \alpha_k^n \mathbf{E}[V(y_k Z^{(k)})], \tag{36}$$

once again by Fatou's lemma. Combining inequalities (34) and (36), we get

$$\begin{aligned} x\gamma_n + \mathbf{E}\Big[V\Big(\gamma_n \frac{d\mathbf{R}^n}{d\mathbf{P}}\Big)\Big] &\leq \sum_{k=n}^{\infty} \alpha_k^n (x y_k + \mathbf{E}[V(y_k Z^{(k)})]) \\ &= \sum_{k=n}^{\infty} \alpha_k^n u_k(x) \leq xy + \mathbf{E}[V(yZ)] = u(x), \end{aligned} \tag{37}$$

where we have used representation (9) for the optimization problem in $[0,T]$ as well as for the ones in $[0,\tau_k]$, $k \in \mathbb{N}$. It is now easy to extend this formula from $\left(\gamma_n, \frac{d\mathbf{R}^n}{d\mathbf{P}}\right)$ to $\left(\gamma, \frac{d\mathbf{R}}{d\mathbf{P}}\right)$, in this way:

$$x\gamma + \mathbf{E}\Big[V\Big(\gamma \frac{d\mathbf{R}}{d\mathbf{P}}\Big)\Big] \leq \lim_n \Big(x\gamma_n + \mathbf{E}\Big[V\Big(\gamma_n \frac{d\mathbf{R}^n}{d\mathbf{P}}\Big)\Big]\Big) \leq u(x). \tag{38}$$

We use these inequalities to prove the following proposition, which is a fundamental step in the direction of our main theorem. Here we consider only the interesting case, i.e. the absolutely-continuous one.

Proposition 4.2 *Under the hypothesis of Theorem 2.8, the following assertions hold true:*

(i) The sequence $(g_n)_n$ is $\boldsymbol{P}$-uniformly integrable;

(ii) $\boldsymbol{R}^n \in \mathcal{M}^a(S^{\tau_n})$ and $\boldsymbol{R} \in \mathcal{M}^a(S^{\tau-})$;

(iii) $g = yZ = u'(x)\dfrac{d\hat{\boldsymbol{Q}}_y}{d\boldsymbol{P}}$.

Proof:*(i)* Recall that, under the Inada conditions, the RAE condition on the limit to $-\infty$ can be given in terms of the function V (see [9, Proposition 4.1]): there is $\zeta_0 > 0$ and $C > 0$ such that

$$\zeta V'(\zeta) \leq C V(\zeta), \quad \text{for}\, \zeta > \zeta_0. \tag{39}$$

Let us fix $K > 0$ constant and consider the quantity $\mathbf{E}[g_n; g_n \geq K]$. If $K > \zeta_0$ and $V'(K) > 0$, from (39) we get $g_n \leq \frac{CV(g_n)}{V'(g_n)} \leq \frac{CV(g_n)}{V'(K)}$, $\forall g_n \geq K$, where the last inequality holds because V' is increasing. In this case we have $\mathbf{E}[g_n; g_n \geq K] \leq \frac{C}{V'(K)}\mathbf{E}[V(g_n); g_n \geq K]$ and it is sufficient to prove the uniform boundedness of $\mathbf{E}[V(g_n); g_n \geq K], n \in \mathbb{N}$, to obtain the uniform integrability of $(g_n)_n$. Indeed, if $\mathbf{E}[V(g_n); g_n \geq K] < \eta \; \forall n \in \mathbb{N}$, for any $\epsilon > 0$ we clearly find a constant $K = K_\epsilon$ sufficiently large such that $\frac{C\eta}{V'(K)} < \epsilon$. Now, since the function V is continuous and strictly convex on $\mathbb{R}_+$ with $V(0) = U(\infty) < \infty$ (we are in the absolutely-continuous case), it is bounded on $[0,K]$. Therefore, proving $(\mathbf{E}[V(g_n); g_n \geq K])_n$ bounded or $(\mathbf{E}[V(g_n)])_n$ bounded is equivalent. On the other hand, by (37) we have

$$\mathbf{E}[V(g_n)] = x\gamma_n + \mathbf{E}[V(g_n)] - x\gamma_n \leq u(x) - x\gamma_n$$

and, since $0 \leq \gamma_n \leq \xi < \infty$, the desired result follows.

(ii) Since S is assumed to be locally bounded, there exists $(\sigma_m)_{m\in\mathbb{N}}$ increasing sequence of stopping time such that $\sigma_m \uparrow \infty$ and $|S^{\sigma_m}| \leq C_m$ $\mathbf{P}$-a.s., for some C_m constant, $\forall m \in \mathbb{N}$. We now show that, $\forall\, n, m \in \mathbb{N}$, $S^{\sigma_m \wedge \tau_n}$ is a $\mathbf{R}^n$-martingale and

$S^{\sigma_m\wedge\tau-}$ is a **R**-martingale. Let us fix $n,m\in\mathbb{N}$ and $0\le s\le t<T$ (or, eventually, permit $t=T$ if $T<\infty$). Since $(y_kZ^{(k)})_k$ is uniformly integrable from *(i)*, we have

$$\mathbf{E}_{\mathbf{R}^n}[S_t^{\sigma_m\wedge\tau_n}|\mathcal{F}_s]=\frac{\mathbf{E}[g_nS_t^{\sigma_m\wedge\tau_n}|\mathcal{F}_s]}{\mathbf{E}[g_n|\mathcal{F}_s]}=\frac{\mathbf{E}[\lim_p\sum_{k=n}^p\alpha_k^ny_kZ^{(k)}S_t^{\sigma_m\wedge\tau_n}|\mathcal{F}_s]}{\mathbf{E}[\lim_p\sum_{k=n}^p\alpha_k^ny_kZ^{(k)}|\mathcal{F}_s]}$$

$$=\frac{\sum_{k=n}^\infty\alpha_k^ny_k\mathbf{E}[Z^{(k)}S_t^{\sigma_m\wedge\tau_n}|\mathcal{F}_s]}{\sum_{k=n}^\infty\alpha_k^ny_k\mathbf{E}[Z^{(k)}|\mathcal{F}_s]}=\frac{\sum_{k=n}^\infty\alpha_k^ny_kZ_s^{(k)}S_s^{\sigma_m\wedge\tau_n}}{\sum_{k=n}^\infty\alpha_k^ny_kZ_s^{(k)}}=S_s^{\sigma_m\wedge\tau_n},$$

by the fact that $S_t^{\sigma_m\wedge\tau_n}$ bounded implies $(y_kZ^{(k)}S_t^{\sigma_m\wedge\tau_n})_k$ and $(\sum_{k=n}^p\alpha_k^ny_kZ^{(k)}S_t^{\sigma_m\wedge\tau_n})_n$ uniformly integrable, for any $1\le n\le p<\infty$ we fix. Here we have used the L^1-convergence of uniformly integrable sequences converging in probability and, in a similar way, we also obtain

$$\mathbf{E}_{\mathbf{R}}[S_t^{\sigma_m\wedge\tau-}|\mathcal{F}_s]=\frac{\mathbf{E}[gS_t^{\sigma_m\wedge\tau-}|\mathcal{F}_s]}{\mathbf{E}[g|\mathcal{F}_s]}=\frac{\mathbf{E}[\lim_n g_nS_t^{\sigma_m\wedge\tau_n}|\mathcal{F}_s]}{\mathbf{E}[\lim_n g_n|\mathcal{F}_s]}$$

$$=\lim_n\frac{\mathbf{E}[g_nS_t^{\sigma_m\wedge\tau_n}|\mathcal{F}_s]}{\mathbf{E}[g_n|\mathcal{F}_s]}=S_s^{\sigma_m\wedge\tau-},$$

as claimed.

(iii) As was emphasized in Remark 2.7, the equivalence of the optimal problem in $[0,T]$ to the one in $[0,\tau[$ holds true and, in particular, $u_{\tau-}(x)=u(x)$, $\forall\,x\in\mathbb{R}$. Hence, using *(ii)* together with (38), the optimality of **R** immediately follows:

$$\mathbf{R}=\hat{\mathbf{Q}}_y,\quad \gamma=y=u'(x)\quad\text{and then}\quad g=yZ, \tag{40}$$

by formulation (13) of the original problem. What we have proved is that every convergent sequence $\left(\gamma_n\frac{d\mathbf{R}^n}{d\mathbf{P}}\right)_n$ of convex combinations of $\{y_nZ^{(n)},n\in\mathbb{N}\}$, admits yZ as limit. More precisely, by statement *(i)*, as $n\to\infty$ we have $\gamma_n\to y$, $\dfrac{d\mathbf{R}^n}{d\mathbf{P}}\overset{\mathbf{P}}{\to}\dfrac{d\hat{\mathbf{Q}}_y}{d\mathbf{P}}$ and

$$x\gamma_n+\mathbf{E}\Big[V\Big(\gamma_n\frac{d\mathbf{R}^n}{d\mathbf{P}}\Big)\Big]\to xy+\mathbf{E}[V(yZ)]=u(x). \tag{41}$$

This proves the last assertion of the proposition and concludes the proof. □

4.2 The proof of the main theorem

All the arguments and results illustrated up to here provide the basis to prove the main results of this paper, stated in Theorem 2.8.

Proof of Theorem 2.8: The first statement follows from both (37) and (41), since$(u_n)_n$ increasing and $u_n\le u$.

(ii) It will first be shown that

$$y_n \frac{d\mathbf{Q}^{(n)}}{d\mathbf{P}} \xrightarrow{\mathbf{P}} y \frac{d\hat{\mathbf{Q}}}{d\mathbf{P}}.$$

In this purpose, it is sufficient to prove that $(y_n Z^{(n)})_n$ is a sequence with the property to be "Cauchy in probability". We use the fact that the function V is strictly convex and then uniformly strictly convex on compacts:
$\forall a > 0,\ K \in \mathbb{R}_+\ \exists \beta > 0$ s. t. $\forall\, \delta_1,\ \delta_2$ with $\delta_1 \in [0, K]$ and $|\delta_1 - \delta_2| \geq a$

$$\frac{V(\delta_1) + V(\delta_2)}{2} > V\Big(\frac{\delta_1 + \delta_2}{2}\Big) + \beta. \tag{42}$$

Suppose that $(y_n Z^{(n)})_n$ is not "Cauchy in probability", i.e. there exists $\alpha > 0$ s. t. $\forall N \in \mathbb{N}\ \exists\, m = m_N, p = p_N\ > N$ with

$$\mathbf{P}\{|y_m Z^{(m)} - y_p Z^{(p)}| > \alpha\} > \alpha. \tag{43}$$

Moreover, since $(y_n Z^{(n)})_n$ is uniformly integrable, there exists $K > 0$ such that

$$\mathbf{P}\{Z^{(n)} > K\} < \frac{\alpha}{2}, \forall\, n \in \mathbb{N}.$$

Let us fix $N \in \mathbb{N}$ and $m, p > N$ satisfying (43), and define the sets

$$\tilde{\Omega} = \{\omega \in \Omega : |y_m Z^{(m)} - y_p Z^{(p)}| > \alpha\}, \Omega_m = \{\omega \in \Omega : Z^{(m)} \leq K\}, \tilde{\Omega}_m = \tilde{\Omega} \cap \Omega_m,$$

It immediately follows that $\mathbf{P}(\tilde{\Omega}) > \alpha,\ \mathbf{P}(\Omega_m) \geq 1 - \alpha/2$ and $\mathbf{P}(\tilde{\Omega}_m) \geq \alpha/2$. Since in $\tilde{\Omega}_m$ we have that (42) holds true, we get

$$\begin{aligned} &x\big(\frac{y_m + y_p}{2}\big) + \mathbf{E}\big[V\big(\frac{y_m Z^{(m)} + y_p Z^{(p)}}{2}\big)\big] \leq \\ &x\big(\frac{y_m + y_p}{2}\big) + \mathbf{E}\Big[\frac{V(y_m Z^{(m)}) + V(y_p Z^{(p)})}{2} 1_{\tilde{\Omega}_m^c}\Big] + \mathbf{E}\Big[\big(\frac{V(y_m Z^{(m)}) + V(y_p Z^{(p)})}{2} - \beta\big) 1_{\tilde{\Omega}_m}\Big] \\ &= x\big(\frac{y_m + y_p}{2}\big) + \mathbf{E}\Big[\frac{V(y_m Z^{(m)}) + V(y_p Z^{(p)})}{2}\Big] - \beta \mathbf{P}(\tilde{\Omega}_m) \\ &\leq \frac{1}{2}\big[x y_n + \mathbf{E}[V(y_n Z^{(n)})] + x y_p + \mathbf{E}[V(y_p Z^{(p)})]\big] - \beta \frac{\alpha}{2}. \end{aligned}$$

Hence, putting

$$\eta_N = \frac{y_m + y_p}{2}, \quad \eta_N \frac{d\mathbf{M}^N}{d\mathbf{P}} = \frac{y_m Z^{(m)} + y_p Z^{(p)}}{2},$$

we have

$$\limsup_{N \to \infty}\ x \eta_N + \mathbf{E}\Big[V\Big(\eta_N \frac{d\mathbf{M}^N}{d\mathbf{P}}\Big)\Big] \leq xy + \mathbf{E}[V(yZ)] - \beta \frac{\alpha}{2}.$$

Now, eventually passing to a convergent sequence $\left(\bar{\gamma}_k \frac{d\bar{\mathbf{Q}}^k}{d\mathbf{P}}\right)_k$ of convex combinations of $\left\{\eta_N \frac{d\mathbf{M}^N}{d\mathbf{P}},\, N \in \mathbb{N}\right\}$ (if $\left(\eta_N \frac{d\mathbf{M}^N}{d\mathbf{P}}\right)_{N\in\mathbb{N}}$ results not to be convergent), for any k in $\mathbb{N}$ we get

$$x\bar{\gamma}_k + \mathbf{E}\Big[V\Big(\bar{\gamma}_k \frac{d\bar{\mathbf{Q}}^k}{d\mathbf{P}}\Big)\Big] \leq xy + \mathbf{E}[V(yZ)] - \beta\frac{\alpha}{2} < u(x),$$

with the same arguments used to obtain (37). On the other hand, by (41), we have

$$x\bar{\gamma}_k + \mathbf{E}\Big[V\Big(\bar{\gamma}_k \frac{d\bar{\mathbf{Q}}^k}{d\mathbf{P}}\Big)\Big] \underset{k}{\to} u(x),$$

in contradiction with the precedent inequalities. This proves that $(y_n Z^{(n)})_n$ is "Cauchy in probability" and then it also converges in probability. From the uniform integrability, this limit also holds in the $L^1(\mathbf{P})$-sense and, by Proposition 4.2, it equals yZ. What we have shown is the convergence

$$y_n Z^{(n)} \overset{L^1(\mathbf{P})}{\longrightarrow} yZ \quad \text{or} \quad Z^{(n)} \overset{L^1(\mathbf{P})}{\longrightarrow} Z$$

and, by (7) and (22), we also have

$$X_T^{(n)}(x) \overset{\mathbf{P}}{\longrightarrow} \hat{X}_T(x),$$

as claimed. □

Since $\hat{X}_T(x)$ is the solution to the problem (1), we know that there exists a sequence $(K^n)_{n\in\mathbb{N}}$ of admissible strategies such that $\mathbf{E}[U(x + (K^n \cdot S)_T)]$ converges to $\mathbf{E}[U(\hat{X}_T(x))]$. In addition to this, Theorem 2.8 provides a sequence $(H^n)_n$ of strategies, as defined in (31), which are not necessarily admissible, but such that $U(x + (H^n \cdot S)_T) \overset{L^1(\mathbf{P})}{\longrightarrow} U(\hat{X}_T(x))$. Moreover, using this result in conjunction with representation (31), we get

$$\hat{X}_T(x) = x + \lim_{n\to\infty} (H^n \cdot S)_{\tau_n},$$

when the limit is taken in probability.

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Beatrice Acciaio
Department of Economy, Finance and Statistics
University of Perugia
Via Pascoli 20, 06123 Perugia, Italia
beatrice.acciaio@stat.unipg.it