On the minimization of area among chord-convex sets

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Abstract. In this paper we study the problem of minimizing the area for the chord-convex sets of given size, that is, the sets for which each bisecting chord is a segment of length at least 2. This problem has been already studied and solved in the framework of convex sets, though nothing has been said in the non-convex case. We give here the relevant definitions and show some first properties.

Keywords. Area-minimizing sets, Chord convex.

1. Introduction and setting of the problem

Consider the convex planar sets with the property that all the bisecting chords (i.e., the segments dividing the set in two parts of equal area) have length at least 2. A very simple question is which is the set in this class which minimizes the area. Surprisingly enough, the answer is not the unit disk, as one would immediately guess, but the so-called "Auerbach triangle", shown in Figure 1 left.

The story of this problem is quite old: already in the 1920's Zindler posed the question whether the disk is the unique planar convex set having all the bisecting chords of the same length (see [4]), while few years later Ulam asked if there are other planar convex sets, besides the disk, which have the floating property, that can be described as follows. Assume that the set has density $1/2$ and it is immersed in the water (hence, half of the set remains immersed while the other half stays out of the water): the set is said to have the "Ulam floating property" if the floating position is of equilibrium, and if this remains true after an arbitrary rotation of the set. For instance, of course the disk has the Ulam floating property, while any other ellipsis does not: indeed, only two floating position of the cilinder are of equilibrium (those for which the water is parallel to one of the two symmetry axes of the ellipsis), while all the other positions are not of equilibrium. In the 1930's, Auerbach

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showed (see [1]) that the two problems above are equivalent, and that there is a whole class of sets having these two properties (we call such sets "Zindler sets", and for instance the Auerbach triangle is a Zindler set). In his paper, Auerbach considered also the question of which Zindler set minimizes the area, among those for which the length of every bisecting chord is the same, say 2: he was able to show that the answer is not given by the disk, and he conjectured that the solution should have been the triangle that has then been called Auerbach triangle, whose area is ≈ 3.11, thus just more or less 1% less than that of the unit disk.

In the last years the problem addressed above was finally solved in [3, 2]. In particular, Fusco and the second author proved in [3] the Auerbach conjecture, that is, the Auerbach triangle minimizes the area among the Zindler sets. Then, Esposito, Ferone, Kawohl, Nitsch and Trombetti in [2] proved that the convex set with minimal area (among those with all the bisecting chords of length at least 2) must be a Zindler set, and thus it is the Auerbach triangle.

Up to now, nothing has been said in the non-convex case, and the aim of this paper is to start working on this more general problem. We can immediately notice that the Auerbach triangle is no longer the solution if we allow other sets to be considered: for instance, consider the "Zindler flower", shown in Figure 1 right. As it appears evident from the figure, the boundary of this set is contained in the union of three equal arcs of circle, each of which covers an angle of 120°. It can be easily calculated that the area of this set is ≈ 2.54, then much smaller than both the area of the unit disk, and that of the Auerbach triangle.

In this paper, we consider the class of the "chord-convex sets", see Definition 2.1: roughly speaking, these sets are not necessarily convex, but have the property that all the bisecting chords are actually segments. We will be able to prove some preliminary interesting properties of these sets, and also to give some counterexamples.

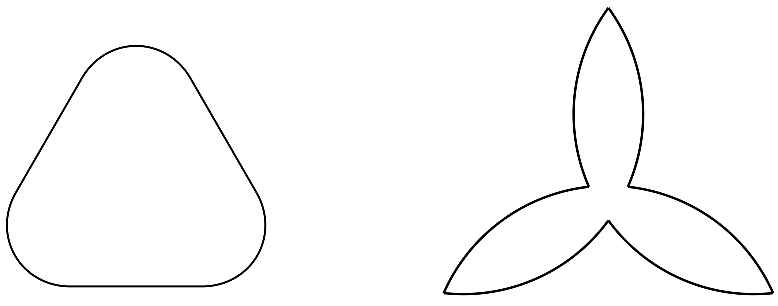

FIGURE 1. The "Auerbach triangle" and the "Zindler flower".

2. Definitions and results

In this section we give all the relevant definitions and we prove our results: it would be impossible to give first the definitions and then present and prove the results, because most of the definitions would not make sense if some properties have not been preliminarly proved: hence, we have to give definitions and results together. We divide these definitions and results in three groups: the uniqueness of the bisecting chords, the properties of the extremes of the bisecting chords and of the intersections between chords, and the Zindler sets. First of all, we can define the class of sets that we will consider.

Definition 2.1. Let $E \subseteq \mathbb{R}^2$ be an open set of finite measure with the property that $E = \mathrm{Int}\,\overline{E}$. The line r is called a *bisecting line* if the intersection of E with each of the two half-spaces in which $\mathbb{R}^2$ is subdivided by r has area exactly $|E|/2$. The set E is called *chord-convex* if the intersection of $\overline{E}$ with each bisecting line is a closed segment, which will be called *bisecting chord*. The *size* of a chord-convex set is the minimal length of a bisecting chord.

Through all the paper, we will always consider chord-convex sets of size at least 2. The main problem that one wants to consider is the minimization of the area among the chord-convex sets of size at least 2. For instance, the unit disk of area π is such a set, as well as the Auerbach triangle, which has area ≈ 3.11, and as the Zindler flower, which has area ≈ 2.54: the first two sets are also convex, while the third is only chord-convex.

2.1. Uniqueness of the bisecting chords of given direction

The first property that we want to investigate is the uniqueness of the bisecting chord of a given direction, to which we will devote the present section. Indeed, it is obvious by continuity that for every direction there is some bisecting chord of that direction, but the uniqueness is not clear, since it is not obvious that a chord-convex set is connected. Actually, as we will see in Theorem 2.6 and Example 2.1, the closure of a chord-convex set is always connected (and even simply connected), but a chord-convex set need not to be connected. However, the uniqueness of the bisecting chord of any given direction is ensured by Theorem 2.6. Before proving that, we need a couple of technical results and of definitions. Throughout this section, E will always denote a chord-convex set.

Definition 2.2. Let x, $y \in \overline{E}$. We say that x and y are *connected* if there is a path in $\overline{E}$ connecting x and y. If b is a bisecting chord, we say that x and b are connected if there is some $y \in b$ such that x and y are connected. Notice that, by definition, if x and b are connected then x is connected to every point $y \in b$.

Lemma 2.3. *Let E be a chord-convex set, and b be a bisecting chord of direction $\bar\theta \in \mathbb{S}^1$, and let $x \in E \cap b$. Then there exists $\varepsilon > 0$ such that for all $\theta \in (\bar\theta - \varepsilon, \bar\theta + \varepsilon)$ there is a unique bisecting chord $b(\theta)$ of direction θ, and* *this chord is connected to x. Moreover, the distance between the chords b and $b(\theta)$ is arbitrarily small, as soon as ε has been chosen small enough.*

Proof. Since E is open and $x \in E$, we can take a square centered in x, with two sides parallel to r, entirely contained in E. Let now $x^{\pm}$ be two points in the interior of the square, contained in the two opposite halfspaces defined by the bisecting line r containing the chord b, and let also $r^{\pm}$ be the two lines parallel to r passing through $x^{\pm}$. The situation is depicted in Figure 2. Let us now define H_r the halfspace "below" the line r, that is, the

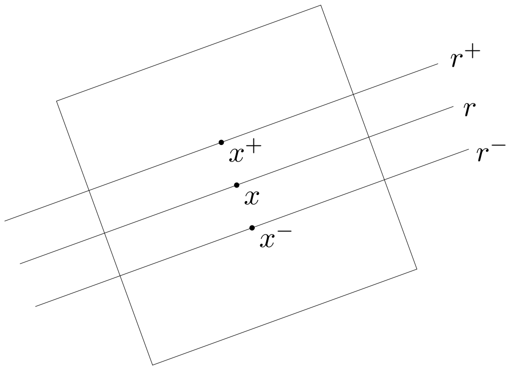

FIGURE 2. The situation in Lemma 2.3.

one containing r^-. We also call H_γ the halfspace "below" γ: this makes sense for every line γ with a direction close to that of r. Since r is a bisecting line, we have $|E \cap H_r| = 1/2$, thus $|E \cap H_{r^-}| < 1/2 < |E \cap H_{r^+}|$ because the square is entirely contained in E. By continuity, there is $\varepsilon > 0$ such that $|E \cap H_{r^-(\theta)}| < 1/2 < |E \cap H_{r^+(\theta)}|$ for all $\theta \in (\bar\theta - \varepsilon, \bar\theta + \varepsilon)$, where $r^+(\theta)$ and $r^-(\theta)$ are the lines of direction θ passing through x^+ and x^- respectively. By continuity, and recalling again that the square is contained in E, we deduce that there is a unique bisecting line of direction θ, which lies between $r^+(\theta)$ and $r^-(\theta)$, and thus intersects the square. Since E is chord-convex, we deduce the existence and uniqueness also of a bisecting chord $b(\theta)$. Since this chord intersects the square, we derive also the closeness of $b(\theta)$ to b, as well as the fact that $b(\theta)$ is connected to x. □

Lemma 2.4. *Let r, s be two different bisecting lines of a chord-convex set E, let T be one of the four corresponding open regions, and let $x \in T \cap E$. Then, there exists a bisecting line passing through x, whose direction belongs to the open interval in $\mathbb{S}^1$ corresponding to T.*

Proof. Let us call, as in Figure 3, r_x and s_x the two lines passing through x and parallel to r and s respectively, and let us also denote, as in the proof of Lemma 2.3, by H_γ the half-space "below" γ for any line γ having direction between those of r and of s. Then, by construction $|E \cap H_{r_x}| \leq |E \cap H_r| = 1/2$,

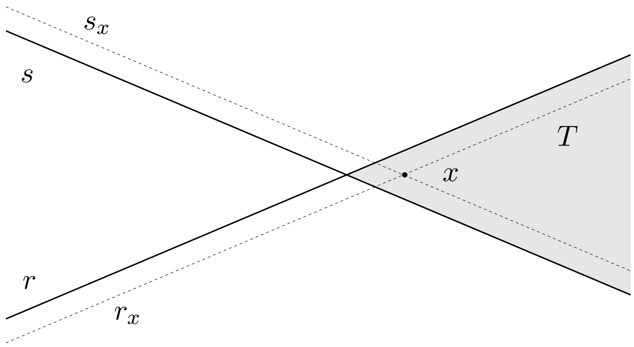

FIGURE 3. The situation in Lemma 2.4.

since $x \in T$ and r is a bisecting line. Moreover, since x belongs to E, then in fact it must be $|E \cap H_{r_x}| < 1/2$, and in the very same way $|E \cap H_{s_x}| > 1/2$. By continuity, there is clearly a line passing through x, with direction in the open interval of $\mathbb{S}^1$ corresponding to T, and which is bisecting. □

Definition 2.5. We say that the sequence of lines $\{r_n\}_{n\in\mathbb{N}}$ *converges* to the line r if the directions of r_n converge in $\mathbb{S}^1$ to the direction of r, and for any ball B big enough the segments $r_n \cap B$ converge in the Hausdorff sense to the segment $r \cap B$.

We are finally in position to prove the main result of this section.

Theorem 2.6. *Let E be a chord-convex set. Then, $\overline{E}$ is connected and simply connected, and there is a unique bisecting chord for every direction in $\mathbb{S}^1$.*

Proof. We will divide the proof of this result in four steps, for the sake of simplicity.

Step I. Every sequence of bisecting lines converge to a bisecting line up to a subsequence.

Let $\{r_n\}$ be a sequence of bisecting lines: first of all, up to a subsequence we can assume that the directions of the lines r_n converge to some $\theta \in \mathbb{S}^1$. Then, we will obtain the existence of a line r such that the sequence $\{r_n\}$ converge to r (up to a subsequence, of course) as soon as we show the existence of a ball B which has non-empty intersection with all the lines r_n. Let then B be a ball centered at the origin and with the property that $|E \cap B| > |E|/2$, which clearly exists since this is true if the radius of the ball is big enough. We have that $B \cap r_n \neq \emptyset$, which follows immediately from the fact that r_n is a bisecting line and thus, as pointed out above, we derive the existence of a line r such that a suitable subsequence of $\{r_n\}$ converges to r. To conclude this step, we only have to show that r is a bisecting line, but this is in turn obvious by continuity and since so are all the lines r_n.

Step II. If r and s are two bisecting lines such that there exists $x \in E \cap r \setminus s$, then for each of the four open regions T_i, $1 \le i \le 4$ determined by r and s one has $|T_i \cap E| > 0$.

Let us assume, without loss of generality, that x belongs to the closures of T_1 and T_2, and that T_3 (resp., T_4) is the region opposite to T_1 (resp. T_2). Then we have by construction that $|E \cap T_1| = |E \cap T_3|$, and in turn $|E \cap T_1| > 0$ because x belongs to the open set E. The same argument shows also $|E \cap T_2| = |E \cap T_4| > 0$, then the step is completed.

Step III. The set $\overline{E}$ is connected and there is a unique bisecting chord for each direction.

Take a generic point $x \in E$, and let r be a bisecting line passing through x. Without loss of generality, let us assume that the line r is horizontal. Define now

$$\bar\theta := \max\Big\{\nu \in [0,\pi] : \forall\, 0 \leq \theta < \nu,\ \exists!\,\text{bisecting chord } b(\theta) \text{ of direction } \theta, \text{ and } b(\theta) \text{ is connected to } x\Big\}.$$

Of course, if we show that $\bar\theta = \pi$ then we have proved at once the uniqueness of the bisecting chords for any direction, and the connectedness of $\overline{E}$ (since any two points of $\overline{E}$ are connected to x, and then they are connected among themselves).

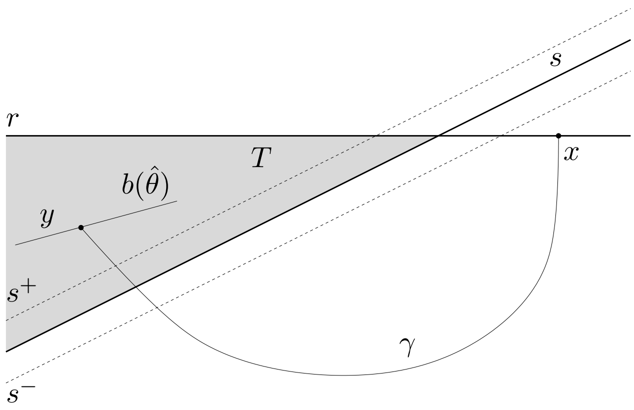

FIGURE 4. The situation in Step III: the region T is shaded.

Let us then assume by contradiction that $\bar\theta < \pi$, and notice that Lemma 2.3 ensures that $\bar\theta > 0$. Pick now a bisecting line s of direction $\bar\theta$ –we still do not know whether this bisecting line is unique, or connected to x, but the existence is obvious. The point x cannot belong to s, because otherwise Lemma 2.3 would give a contradiction to the maximality of $\bar\theta$. Let us call T the region (shaded in Figure 4) determined by r and s corresponding to the angles between 0 and $\bar\theta$ and not containing x in its closure. By Step II we have $|T \cap E| > 0$, so in particular there is some point $y \in T \cap E$. Applying Lemma 2.4 to this point y and the region T, we find a bisecting line passing through y and with direction $\hat\theta \in (0, \bar\theta)$. By definition of $\bar\theta$, we know that this line is the unique bisecting line with direction $\hat\theta$, and that its intersection $b(\hat\theta)$

with $\overline{E}$ is connected to x. Thus, there is some path $\gamma \subseteq \overline{E}$ starting from y and ending at x, and this path must clearly intersect s.

As in Figure 4, let us call $s^\pm$ two lines parallel to s, lying on the opposite hyperplanes defined by s. We can choose the lines very close to s, so in particular neither y nor x is between them, and hence γ must cross both $s^\pm$. Recalling that γ is contained in $\overline{E}$, and calling again $H_{s^\pm}$ the half-space "below" $s^\pm$, we deduce

$$\left|E \cap H_{s^-}\right| < \left|E \cap H_s\right| = \frac{1}{2} < \left|E \cap H_{s^+}\right|.$$

Arguing exactly as in Lemma 2.3, we obtain then the existence and uniqueness of a bisecting line of direction θ for any $\theta \in (\bar\theta - \varepsilon, \bar\theta + \varepsilon)$, and the corresponding bisecting chord $b(\theta)$ must surely be connected to x, since it intersects the path γ. Since this is in contrast with the definition of $\bar\theta$, we have obtained $\bar\theta = \pi$ which –as noticed above– concludes this step.

Step IV. The set $\overline{E}$ is simply connected.

To conclude the proof of the theorem, we only need to check that the set $\overline{E}$ is simply connected. If this were not true, there would exist a closed curve $\gamma \subseteq \overline{E}$ enclosing some small ball $B \subseteq \mathbb{R}^2 \setminus \overline{E}$. Pick any point $x \in B$, and take any bisecting line r passing through x: by construction, each of the two halflines contained in r and having x as endpoint intersect γ, thus $\overline{E}$. Since this implies that $r \cap \overline{E}$ is not a segment, the contradiction comes from the fact that E is chord-convex. □

We can immediately observe two simple consequences of the above Theorem.

Corollary 2.7. *The claim of Lemma* 2.4 *is valid for any $x \in T$, not only for the points $x \in T \cap E$.*

Proof. Let us call, as in the proof of Lemma 2.4, H_r, H_s, H_{r_x} and H_{s_x} the four half-spaces determined by the lines r and s, and by the lines r_x and s_x parallel to r and s but passing through x.

Again, we know that $|E \cap H_{r_x}| \leq |E \cap H_r| = 1/2$ since $H_{r_x} \subseteq H_r$. There are now two possibilities: either $|E \cap H_{r_x}| = 1/2$, or $|E \cap H_{r_x}| < 1/2$. The first case can be excluded because otherwise r and r_x would be two different parallel bisecting lines, which is impossible by Theorem 2.6; then, $|E \cap H_{r_x}| < 1/2$, and in the very same way $|E \cap H_{s_x}| > 1/2$. The conclusion follows then exactly as in Lemma 2.4. □

Corollary 2.8. *In any chord-convex set E, every two bisecting chords intersect.*

Proof. Suppose that there exists two bisecting lines, r and s, such that the corresponding bisecting chords $b(r)$ and $b(s)$ do not intersect. As in Figure 5, let us then call T and $\widetilde{T}$ two of the regions in which r and s divide the plane, so that $b(r)$ and $b(s)$ belong to the closure of $\widetilde{T}$, and T is opposite to $\widetilde{T}$. Since both r and s are bisecting lines, we have $|T \cap E| = |\widetilde{T} \cap E|$.

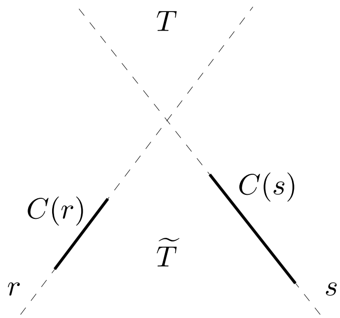

FIGURE 5. Situation in Corollary 2.8.

Now, if $|T \cap E| > 0$ then there is some $x \in T \cap E$, but this is impossible because this x would not be connected with the two chords: indeed, a curve connecting x with $b(r)$ should somewhere exit from the region T, and this would happen at some point in $r \setminus b(r)$, or in $s \setminus b(s)$, against the definition of bisecting chords. On the other hand, if $|T \cap E| = 0$, then also $|\widetilde{T} \cap E| = 0$, and we run into the same contradiction, because then the two chords $b(r)$ and $b(s)$ would not be connected with each other. □

In the above Theorem 2.6, to get the simply connectedness it was necessary to consider the closure $\overline{E}$ of E. In fact, there exist chord-convex sets which are not simply connected (but their closure is of course simply connected, by Theorem 2.6). An example is shown below.

Example. Let us present now an example of a chord-convex set which is not simply connected. As in Figure 6, let AD be a segment, and let the points C and B divide it in three equal parts. Then, let P and Q be two points on the circle centered at C and passing through A and B, such that the segment PQ passes through C. Analogously, let R and S be two points on

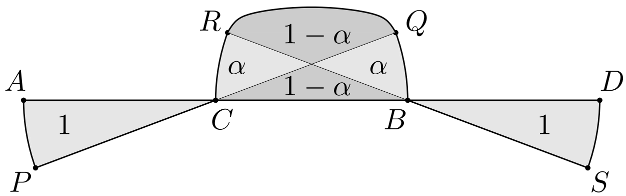

FIGURE 6. Example 2.1: the set E is shaded.

the circle centered at B passing through C and D, so that the segment RS passes through B. Let finally E be the bounded set whose boundary is the union of the segments AC, BD, PC and BS, the arcs of circle $\widehat{AP}$, $\widehat{BQ}$, $\widehat{CR}$ and $\widehat{SD}$, and some convex curve connecting R and Q as in the Figure. Of course, a suitable choice of this curve and the other parameters allows us to consider that the different parts of E as sides either 1, or α, or $1-\alpha$ for some $0<\alpha<1$, as indicated in Figure 6. As a consequence, it is very quick to understand that this set E is chord-convex, but it is not connected (as already said, of course $\overline{E}$ is connected, according to Theorem 2.6).

2.2. Properties of the extreme points and of the intersections between chords

Thanks to Theorem 2.6, we have the uniqueness of the bisecting chords of any direction. This allows us to give the following definition.

Definition 2.9. Let E be a chord-convex set. For any $\theta \in \mathbb{S}^1$, we denote by $r(\theta)$ the unique bisecting line of E with direction θ, and by $L(\theta)$, $M(\theta)$ and $R(\theta)$ the left extreme, the center and the right extreme, respectively, of the corresponding bisecting chord, that we denote by $b(\theta)$. As a consequence, $b(\theta) = r(\theta) \cap \overline{E} = [L(\theta), R(\theta)]$.

Notice that of course

$$M(\theta+\pi) = M(\theta)\,, \qquad L(\theta+\pi) = R(\theta)\,, \qquad R(\theta+\pi) = L(\theta)\,.$$

Moreover, for any $\theta \in \mathbb{S}^1$, we will call π_θ the projection on the bisecting line $r(\theta)$, and define

$$L^+(\theta) := \varlimsup_{\alpha\to\theta} \pi_\theta(L(\alpha))\,, \qquad R^-(\theta) := \varliminf_{\alpha\to\theta} \pi_\theta(R(\alpha))$$

while, by definition,

$$L(\theta) = \varliminf_{\alpha\to\theta} \pi_\theta(L(\alpha))\,, \qquad R(\theta) = \varlimsup_{\alpha\to\theta} \pi_\theta(R(\alpha))\,.$$

Proposition 2.10. *Let E be a chord-convex set with size 2. Then, for every $\theta \in \mathbb{S}^1$,*

$$\overline{L(\theta)R^-(\theta)} \geq 2\,, \qquad \overline{L^+(\theta)R(\theta)} \geq 2\,.$$

Proof. By symmetry, it is enough to show the first inequality. Let us also assume for simplicity of notation that $\theta = 0$, and assume that $\overline{L(0)R^-(0)} < 2$. By definition of R^-, we can find directions ξ arbitrarily close to 0 with $\pi_0(R(\xi)) \leq R^-(0) + \varepsilon$; on the other hand, if ξ is close enough to 0, then $\pi_0(L(\xi)) \geq L(0) - \varepsilon$. Let then $\xi \ll 1$ be a direction for which both the inequalities hold: one has then

$$\overline{L(\xi)R(\xi)} = \frac{\pi_0(R(\xi)) - \pi_0(L(\xi))}{\cos\xi} \leq \frac{\overline{L(0)R^-(0)} + 2\varepsilon}{\cos\xi} < 2\,,$$

where the last inequality is true as soon as both ε and $|\xi|$ are small enough. This gives a contradiction with the fact that the size of E is at least 2, and this concludes the thesis. ☐

Let us now prove that the intersection between any two bisecting chords is always in the “internal part” of both, that is, between L^+ and R^-.

Lemma 2.11. *Let E be a chord-convex set. Then, for any $\theta \neq \xi \in \mathbb{S}^1$, one has*

$$b(\theta) \cap b(\xi) \in [L^+(\theta), R^-(\theta)]\,.$$

Proof. Let us assume for simplicity that $\theta = 0$, and assume also that the claim is false. Hence, there exists some $\xi \neq 0$ such that $b(0) \cap b(\xi) \in [L(\theta), L^+(\theta))$. By definition of L^+, we can find an arbitrarily small α with $L(\alpha)$ very close to $L^+(\theta)$, in particular $\pi_0(L(\alpha)) > b(0) \cap b(\xi)$. Since we can choose such a direction satisfying $|\alpha| < |\xi|$, we obtain that the bisecting chords $b(\alpha)$ and $b(\xi)$ do not intersect, which is absurd by Corollary 2.8. □

We can now deduce that the segments $[L, L^+]$ belong to the boundary of E.

Lemma 2.12. *For any $\theta \in \mathbb{S}^1$, we have $b(\theta) \cap E \subseteq (L^+(\theta), R^-(\theta))$, which in particular implies that the interval $[L(\theta), L^+(\theta)]$ belongs to ∂E.*

Proof. As usual, let us take $\theta = 0$ and write $L = L(0)$ and $L^+ = L^+(0)$ for simplicity of notations. By definition, it is clear that $L \in \partial E$, hence there is nothing to prove if $L = L^+$; moreover, since ∂E is closed, it is enough to exlude the presence of some point of E in the open interval (L, L^+). Suppose then that such a point exists, thus there exists some small ball contained in E and centered at a point of $(L, L^+) \cap E$, as in Figure 7. Any point of

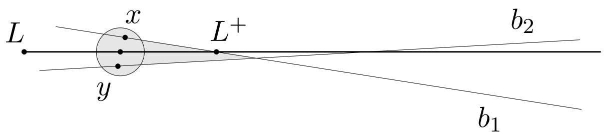

FIGURE 7. The situation in Lemma 2.12.

this ball if of course contained in some bisecting chord; if we take points arbitrarily close to the center, we get that the corresponding bisecting chords become very close to be horizontal: otherwise, we would find a bisecting chord $b(\xi)$ for some $\xi \neq 0$ which intersects $b(0)$ in the center of the ball, against Lemma 2.11.

Let us then take two points of the ball, x and y, respectively above and below $b(0)$, and consider two bisecting chords b_1 and b_2 passing through x and y, which have of course respectively a negative and a positive direction, since they must intersect $b(0)$. Again by Lemma 2.11, we know that both these chords intersect $b(0)$ at some point in $[L^+, R^-]$; since $\overline{E}$ is simply connected by Theorem 2.6, we obtain that E contains the open region $\mathcal{R}$ shaded in Figure 7, which is the union of the ball and the triangle with extremes x, y and the intersection between b_1 and b_2. Since L^+ is the limit of left extremes, it belongs to ∂E: thus, at least one between b_1 and b_2 must pass through L^+, for instance the figure depicts a situation where $L^+ \in b_1$ but $L^+ \notin b_2$.

Let us use again that fact that L^+ is the limit of points $L(\theta_i)$ for a suitable sequence $\theta_i \to 0$. Take some θ_i such that $|\theta_i|$ is smaller than both the directions of b_1 and of b_2, and consider the point $L(\theta_i)$: it cannot belong to $\mathcal{R}$, because it belongs to ∂E while $\mathcal{R} \subseteq E$. On the other hand, if it does not belong to $\partial\mathbb{R}$ then the chord $b(\theta_i)$ cannot intersect both b_1 and b_2, which is absurd: we deduce that $L(\theta_i)$ belongs to one of the two segments xL^+ and yL^+ (in particular, one which is part of b_1 or b_2). However, this also leads to a contradiction, because since $|\theta_i| \ll 1$ then the bisecting line $r(\theta_i)$ would enter inside $\mathcal{R}$ before the left extreme $L(\theta_i)$, which is a contradiction with the definition of bisecting chord. We have thus concluded the proof. $\square$

It is now useful to introduce the further notation.

Definition 2.13. A point $x \in [L(\theta), R(\theta)]$ is said to be *above* E (resp. *below* E) if there is a ball $B(x,\rho)$ of center x and radius ρ such that $B(x,\rho) \cap H_{r(\theta)} \subseteq \overline{E}$ (resp. $B(x,\rho) \cap H^c_{r(\theta)} \subseteq \overline{E}$). An interval $I \subseteq [L(\theta), R(\theta)]$ is said to be above E (resp. below E) if is made of points which are all above E (resp. below E).

Lemma 2.14. *Let E be a chord-convex set and assume that for some $\theta \in \mathbb{S}^1$ there exists a point $z \in [L(\theta), L^+(\theta))$ which is below E. Then the whole segment $(z, L^+(\theta))$ is below E, and there is no point in $[L(\theta), L^+(\theta))$ which is above E.*

Proof. The proof follows with the very same argument as in Lemma 2.12. Indeed, assume as usual that $\theta = 0$ and take a point $z \in [L(0), L^+(0))$ below E: by definition, this means that there is a small ball centered at z whose upper half ball belongs to E. As in the proof of Lemma 2.12, we can take some point x in this ball very close to z, so that a bisecting chord passing through x must have a negative slope close to 0 and cross $b(0)$ in a point which belongs to $[L^+(0), R^-(0)]$. As a consequence, the simply connectedness of $\overline{E}$ given by Theorem 2.6 ensures that the whole open triangle $xzL^+(0)$ is contained in E, and this implies that every point of the segment $(z, L^+(0))$ is below E, which concludes the first part of the proof.

The second part follows immediately: if some point in $[L(0), L^+(0))$ were above E, by the first part there would be points in $[L(0), L^+(0))$ which are at the same time above an below E, hence which are inside E. And in turn, this gives a contradiction with Lemma 2.12, because the whole segment $[L(0), L^+(0)]$ must be in ∂E. $\square$

We can now characterize the intersection $b(\theta) \cap E$ for any angle θ. Let us be more precise: fix for simplicity $\theta = 0$, and write again L, L^+, R^- and R in place of $L(0)$, $L^+(0)$, $R^-(0)$ and $R(0)$. Define then, for any $\theta \neq 0$, $P(\theta)$ as the intersection between $b(0)$ and $b(\theta)$; moreover, define the following points in $[L^+, R^-]$,

$$Q_l^+ := \overline{\lim_{\theta \nearrow 0}} P(\theta)\,, \quad Q_l^- := \underline{\lim_{\theta \nearrow 0}} P(\theta)\,, \quad Q_r^+ := \overline{\lim_{\theta \searrow 0}} P(\theta)\,, \quad Q_r^- := \underline{\lim_{\theta \searrow 0}} P(\theta)\,.$$

Arguing exactly as in Lemma 2.12 and Lemma 2.14, we can prove that

$$\begin{aligned} &(L^+, Q_l^+) \text{ and } (Q_r^-, R^-) \text{ are below } E, \\ &\text{while } (L^+, Q_r^+) \text{ and } (Q_l^-, R^-) \text{ are above } E\,. \end{aligned} \tag{2.1}$$

Indeed, for any $\varepsilon > 0$ we can find some $0 < -\theta \ll 1$ such that $\pi_0(L(\theta)) < L^+ + \varepsilon$ and $\pi_0(P(\theta)) > Q_l^+ - \varepsilon$. Since the function $\beta(\theta) = \pi_0^{-1}(L^+ + \varepsilon) \cap b(\theta)$

is well defined and continuous near 0, we deduce that the vertical segment connecting $L^+ + \varepsilon$ to $\beta(\theta)$ is entirely contained in E, thus again Theorem 2.6 ensures that the triangle of vertices $L^+ + \varepsilon$, $\beta(\theta)$ and $P(\theta)$ is inside E, and in turn this implies that all the points of $(L^+ + \varepsilon, Q_l^+ - \varepsilon)$ are below E, which by letting $\varepsilon \to 0$ implies that the interval (L^+, Q_l^+) is below E. The very same argument shows also the claims about the other three intervals, so (2.1) is established. As a consequence, if we set

$$Q^+ = \min\{Q_l^+, Q_r^+\}, \qquad\qquad Q^- = \max\{Q_l^-, Q_r^-\},$$

we know that E contains the open intervals (L^+, Q^+) and (Q^-, R^-). There are now two possible cases: if $Q^+ > Q^-$, then the whole interval (L^+, Q^-) is inside E. Instead, if $Q^+ \leq Q^-$, then we know only that E contains (L^+, Q^+) and (Q^-, R^-); the points in (Q^+, Q^-) are then all above E but not necessarily below E (if $Q_l^- \leq Q_l^+ = Q^+ < Q^- = Q_r^- \leq Q_r^+$), or all below E but not necessarily above E (if $Q_r^- \leq Q_r^+ = Q^+ < Q^- = Q_l^- \leq Q_l^+$). All these observations become particularly useful in a specific case, namely, if the functions L and R are continuous at $\theta = 0$: indeed, in this case, obviously $L = L^+$ and $R = R^-$, and it easily follows that the four points $Q^\pm_{r,\,l}$ coincide all with the middle point of $b(0)$ (this follows from Lemma 2.16 below). We can then summarize what we found in the following result.

Lemma 2.15. *Let E be a chord-convex set such that the functions L and R are continuous. Then, the interior of any bisecting chord $b(\theta)$ is contained inside E, except possibly the middle point $M(\theta)$.*

Let us now show what we just mentioned, that is, the intersection between bisecting chords converges to their middle point when L and R are continuous.

Lemma 2.16. *Let E be a chord-convex set such that L and R are continuous. Then, for any $\theta \in \mathbb{S}^1$, the point $b(\theta) \cap b(\xi)$ converges to $M(\theta)$ when $\xi \to \theta$.*

Proof. Let us call ℓ the length of the chord $b(\theta)$. For any $\xi \in \mathbb{S}^1$, since both $b(\theta)$ and $b(\xi)$ are bisecting chords then we know that $|T \cap E| = |T' \cap E|$, where T and T' are two opposite regions in which $\mathbb{R}^2$ is divided by the two lines $r(\theta)$ and $r(\xi)$. In particular, let $\xi = \theta + \eta$ be very close to θ, and let T and T' the two opposite "small" regions, that is, those corresponding to the small corner $\eta \ll 1$. Since L and R are continuous, we know that the extremes of any bisecting chord of directions between θ and ξ are at most a distance ε apart from those of $b(\theta)$; as a consequence, if the point $b(\theta) \cap b(\xi)$ is at distances d and $\ell - d$ from the extremes of $b(\theta)$, we have

$$\frac{(d-\varepsilon)^2}{2}\,|\eta| \leq |T \cap E| \leq \frac{(d+\varepsilon)^2}{2}\,|\eta|\,,$$
$$\frac{(\ell-d-\varepsilon)^2}{2}\,|\eta| \leq |T' \cap E| \leq \frac{(\ell-d+\varepsilon)^2}{2}\,|\eta|\,,$$

and it follows that d converges to $\ell/2$ when $\eta \to 0$, that is the thesis. $\square$

We can conclude this section with an important result, which states that the intersection point between any two bisecting chords cannot be an extreme point for both them.

Theorem 2.17. *Let E be a chord-convex set. Then, two bisecting chords cannot intersect at a point which is extreme point for both them.*

Proof. Let us suppose that the claim is false. In particular, we can assume that $L := L(0) = L(\theta)$ for some $0 < \theta < \pi$ (if for such a θ one has $L(0) = R(\theta)$ then a very similar argument would work).

We observe then that, for every $0 < \xi < \theta$, the bisecting chord $b(\xi)$ must pass through L, since it must cross both $b(0)$ and $b(\theta)$. By definition of R^-, we can now take some $0 < \bar{\xi} < \theta$ such that $\pi_0(R(\xi)) > R^- - \varepsilon$ for every $0 < \xi < \bar{\xi}$. As a consequence, for every $0 < \xi < \bar{\xi}$ we have that

$$\overline{LR(\xi)} \geq \overline{L\pi_0(R(\xi))} > R^- - L - \varepsilon > 2 - \varepsilon\,,$$

where the last inequality comes by Proposition 2.10, assuming without loss of generality that the size of E is 2. We call for brevity $\eta := R^- - L - \varepsilon > 0$. Let us now fix any two directions $0 < \xi_1 < \xi_2 < \bar{\xi}$, and call T and T' the two

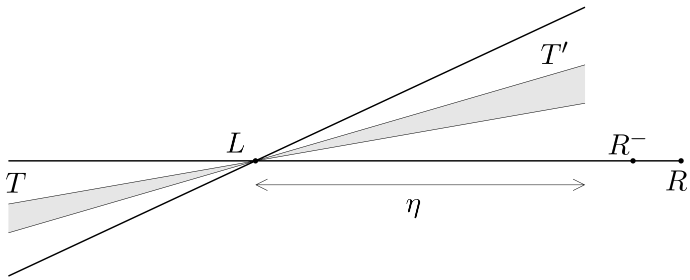

FIGURE 8. The situation in Theorem 2.17.

regions determined by the bisecting lines $r(\xi_1)$ and $r(\xi_2)$, as in Figure 8. By construction we have

$$|E \cap T| = |E \cap T'| \geq \frac{\eta^2(\xi_2 - \xi_1)}{2}\,,$$

and this implies that there is some point $x \in E \cap T$ with $\pi_0(x) < L - \eta$. By construction, a bisecting chord passing through x must have direction between 0 and θ; but then, it must pass through L, and so its direction is actually between ξ_1 and ξ_2. Summarizing, for any choice of $0 < \xi_1 < \xi_2 < \theta$ we have found a direction $\xi \in (\xi_1, \xi_2)$ such that $\pi_0(L(\xi)) \leq L - \eta$. If we now send both ξ_1 and ξ_2 to 0, so does also ξ, and this implies that

$$L = \varliminf_{\xi \to 0} \pi_0(L(\xi)) \leq L - \eta\,,$$

which is absurd. This concludes the proof. □

Notice that the above Theorem does not say that the intersection point of two bisecting chords is always in the interior of both them, but only that it cannot be extreme for both them. For instance, in the case of the Zindler flower of Figure 1, the left extreme $L(\pi/2)$ is the intersection point of $b(\pi/2)$ with $b(0)$: in particular, this point is left extreme of a bisecting chord, and middle point of the other one. We can immediately observe that this is always the case, at least when L and R are continuous functions.

Corollary 2.18. *Let E be a chord-convex set such that L and R are continuous. Then, if the intersection of two bisecting chords is an extreme point of one of them, it must be the middle point of the other one.*

Proof. This immediately follows from Theorem 2.17 and Lemma 2.15. Indeed, assume that $x := b(\theta) \cap b(\xi)$ coincides with $L(\theta)$ for some $\theta \neq \xi \in \mathbb{S}^1$. Theorem 2.17 ensures that x is neither $L(\xi)$ nor $R(\xi)$, hence $x \in (L(\xi), R(\xi))$. On the other hand, $x \in \partial E$, and the only point of $(L(\xi), R(\xi))$ which can be in ∂E is the middle point, according to Lemma 2.15. □

2.3. Zindler sets and their properties

In this last section we define the Zindler sets and we prove their main properties.

Definition 2.19. Let E be a chord-convex set. We say that E is a *Zindler set* if all the bisecting chords have the same length.

As we said in the introduction, the Zindler sets play an important role in the problem of minimizing the area among the *convex* set of given size: roughly speaking, it is very easy to guess (but hard to show!) that a minimizer of the area must be a Zindler set. In fact, the proof of the minimality of the Auerbach triangle was done in two steps, by different authors: it was first proved that the Auerbach triangle minimizes the area among the (convex) Zindler sets, and then that a minimizer among the convex sets (which trivially exists by compactness) must be Zindler.

The general situation of the chord-convex sets that we are considering now seems even more complicated, though there are common points. First of all, it is not obvious whether a minimizer of the area among the chord-convex sets exists, since while the class of convex sets is compact, the class of the chord-convex sets is not so. Moreover, it is again extremely reasonable to guess that, if a minimizer exists, then it must be a Zindler set: we are not able to show this result in full generality, but we can prove a particular case in Theorem 2.23.

We can immediately observe that, for a Zindler set, the functions L and R are automatically continuous, hence in particular Lemma 2.15 and Corollary 2.18 always apply for a Zindler set.

Lemma 2.20. *Let E be a Zindler set. Then, the functions L and R are continuous.*

Proof. This comes readily from Proposition 2.10. Indeed, assuming without loss of generality that the size of E is 2, for any $\theta \in \mathbb{S}^1$ we know on one side that $\overline{L(\theta)R(\theta)} = 2$ because E is a Zindler set, and on the other hand that $\overline{L(\theta)R^-(\theta)} \geq 2$ by Proposition 2.10. It follows that $R^-(\theta) = R(\theta)$, and similarly that $L^+(\theta) = L(\theta)$. By definition of L^+ and R^-, the continuity of L and R is then obvious. □

Recall now that Corollary 2.18 tells us that, for a Zindler set, the intersection between two bisecting chords has two possibilities: either it is an internal point of both chords, or it is at the same time extreme point of one of them, and middle point of the other one. Of course the second case is more peculiar, and we will call "edge angle" any of the two directions. More precisely, it is useful to give the next definition.

Definition 2.21. For any chord-convex set E, the sets $\mathcal{L}$, $\mathcal{R}$, $\mathcal{ML}$ and $\mathcal{MR}$ are defined as

$$\begin{aligned}
\mathcal{L} &:= \Big\{\theta \in \mathbb{S}^1 : \exists\, \eta \in [\theta - \pi/2, \theta + \pi/2] \text{ such that } L(\theta) = M(\eta)\Big\},\\
\mathcal{R} &:= \Big\{\theta \in \mathbb{S}^1 : \exists\, \eta \in [\theta - \pi/2, \theta + \pi/2] \text{ such that } R(\theta) = M(\eta)\Big\},\\
\mathcal{ML} &:= \Big\{\theta \in \mathbb{S}^1 : \exists\, \eta \in [\theta - \pi/2, \theta + \pi/2] \text{ such that } M(\theta) = L(\eta)\Big\},\\
\mathcal{MR} &:= \Big\{\theta \in \mathbb{S}^1 : \exists\, \eta \in [\theta - \pi/2, \theta + \pi/2] \text{ such that } M(\theta) = R(\eta)\Big\}.
\end{aligned}$$

If θ belongs to any of the above sets, we call it an *edge angle.*

We can immediately notice a technical property of the intervals which are contained in one of the above sets. We state it for $\mathcal{ML}$, but of course the analogous results for the other sets are also valid.

Lemma 2.22. *Let E be a chord-convex set such that L and R are continuous, assume that $I \subseteq \mathcal{ML}$ for some interval $I \subseteq \mathbb{S}^1$ and define $\psi : I \to \mathbb{S}^1$ the function such that $M(\theta) = L(\psi(\theta))$ for any $\theta \in I$. Then, the function ψ is decreasing.*

Proof. First of all, observe that the function ψ is well-defined, since by Theorem 2.17 it is not possible that two different directions have the same left extreme. Let us assume without loss of generality that $I = (0, \bar\theta)$ and $0 < \psi(0) < \pi$.

We claim that for any $\theta \in I$ the middle point $M(\theta)$ is below $r(0)$: indeed, since the function ψ is clearly continuous, then $0 < \psi(\theta) < \pi$, and then if $M(\theta) = L(\psi(\theta))$ is above $r(0)$ the two bisecting chords $b(0)$ and $b(\psi(\theta))$ do not intersect, which is absurd by Corollary 2.8. Analogously, since $b(\theta)$ must intersect $b(\psi(0))$, then by construction $b(\theta) \cap b(\psi(0)) \in (M(\theta), R(\theta))$. Finally, since $M(\theta) = L(\psi(\theta))$ and $b(\psi(\theta))$ must intersect $b(\psi(0))$, it follows $0 < \psi(\theta) < \psi(0)$. The monotonicity of the function ψ then follows. □

As we said above, it is reasonable to expect that the intersection of two bisecting chords is usually an internal point for both of them, and that the edge angles are quite rare: for instance, the Zindler flower of Figure 1 has six edge angles (corresponding to three "bad" pairs of chords), and a simple modification –namely, a flower with n petals instead of 3– gives an example with $3n$ edge angles. In particular, if the edge angles are finite or countably many, we can show that a minimizer of the area –if it exists– must be a Zindler set. It is actually enough something even weaker, namely, that the edge angles do not fill any open interval.

Theorem 2.23. *Assume that E minimizes the area among the chord-convex sets of size* 2*. Assume in addition that L and R are continuous, and that the directions which are not edge angles are dense (for instance, this is true if the edge angles have zero length in $\mathbb{S}^1$). Then, E is a Zindler set.*

Proof. Let us assume that E is not a Zindler set: then, there must be a direction such that the corresponding bisecting chord has length strictly more than 2. Actually, the same remains true for all the directions in a small neighborhood, because L and R are continuous, and so is then also the length of the bisecting chords. Since the non-edge angles are dense, we can then assume the existence of a direction (say, 0) which is a non-edge angle and for which the bisecting chord has length $2\ell > 2 + 6a$, for some strictly positive a.

Let us now apply Lemma 2.16 to get the continuity of the function $\tau : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^2$ defined as $\tau(\theta, \xi) = b(\theta) \cap b(\xi)$ for $\theta \neq \xi$, and $\tau(\theta, \theta) = M(\theta)$. Since 0 is not an edge angle, $L(0)$ and $R(0)$ do not belong to the image of τ, hence we can assume, possibly up to decrease a, that

$$\tau(\theta, \xi) \notin B\big(L(0), 6a\big) \cup B\big(R(0), 6a\big) \qquad \forall\, \theta,\, \xi \in \mathbb{S}^1 . \tag{2.2}$$

By the continuity of L and R, there exists $\bar\theta > 0$ such that

$$\max\Big\{\overline{L(\theta)L(0)},\, \overline{R(\theta)R(0)}\Big\} < a \qquad \forall\, \theta \in (-\bar\theta, \bar\theta) . \tag{2.3}$$

Let us now call $x = \tau(-\bar\theta, \bar\theta)$: again by Lemma 2.16, up to decrease $\bar\theta$ we have also

$$\overline{xM(0)} < a . \tag{2.4}$$

Let us now consider the four regions in which $\mathbb{R}^2$ is subdivided by the bisecting lines $r(\bar\theta)$ and $r(-\bar\theta)$, and call T and T' the two small ones, corresponding to the directions between $-\bar\theta$ and $\bar\theta$: as usual, we know that $|T \cap E| = |T' \cap E|$. Putting together (2.3) and (2.4), we have that for any $\theta \in (-\bar\theta, \bar\theta)$ both the points $L(\theta)$ and $R(\theta)$ have distance at least $\ell - 2a$ from x. Thus, recalling that $\overline{E}$ is simply connected by Theorem 2.6,

$$B(x, \ell - 2a) \cap \big(T \cup T'\big) \subseteq E . \tag{2.5}$$

We define now the competitor set

$$\widetilde{E} := \big(E \setminus (T \cup T')\big) \cup \big(B(x, \ell - 3a) \cap (T \cup T')\big) .$$

By (2.5) we know that $\widetilde{E}$ is strictly contained in E, so it has a strictly smaller volume: we will conclude the proof by showing that $\widetilde{E}$ is also a chord-convex set of size at most 2.

First of all, observe that by construction $|T \cap \widetilde{E}| = |T' \cap \widetilde{E}|$, hence

$$\big|(E \setminus \widetilde{E}) \cap T\big| = \big|(E \setminus \widetilde{E}) \cap T'\big|\,.$$

This implies that the lines $r(\bar\theta)$ and $r(-\bar\theta)$ are bisecting lines also for $\widetilde{E}$, and in turn this ensures that, for every $\theta \in (-\bar\theta, \bar\theta)$, the unique bisecting line of direction θ is the one crossing x, whose intersection with $\widetilde{E}$ is a segment of length $2(\ell - 3a) > 2$. To conclude that $\widetilde{E}$ is chord-convex and has size at most 2, it is then sufficient to show that for any $\theta \notin (-\bar\theta, \bar\theta)$ the line $r(\theta)$ is a bisecting line also for $\widetilde{E}$, and its intersection with the closure of $\widetilde{E}$ coincides with $b(\theta)$ (and it is then a segment of length at least 2). Actually, since we already checked that $r(\pm\bar\theta)$ are bisecting lines for $\widetilde{E}$, it is enough to consider directions $\theta \notin [-\bar\theta, \bar\theta]$.

Let then θ be such an angle; notice that the intersection of $r(\theta)$ with the region $T \cup T'$ is a segment PQ, and the points P and Q coincide by definition with $\tau(\theta, \bar\theta)$ and $\tau(\theta, -\bar\theta)$, so they are both in $b(\theta)$ and in $b(\pm\bar\theta)$. By (2.2) and a trivial geometrical argument, both P and Q are inside the ball $B(x, \ell - 3a)$, so the segment PQ is entirely inside $\widetilde{E}$ and the proof is concluded. $\square$

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