Consistency of traded option prices and absence of arbitrage in the presence of stochastic interest rates

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Abstract

We consider a model-independent setting where finitely many call options on a bond, with possibly different maturities, are liquidly traded. We individuate necessary and sufficient conditions for the market prices to be consistent with the existence of a pricing measure, and relate these conditions to the concepts of weak and strong arbitrage. In particular, a new condition arises, with respect to those obtained by Davis and Hobson [6] for the case of call options on a stock in a deterministic-rate framework.

1 Introduction

Starting with the seminal paper [4], substantial research has been done on the relationship between the option prices quoted in the market and the price distributions of the underlying asset, and on the consistency of the option prices with absence of arbitrage opportunities. A widely used assumption is the availability in the market of call options with a given maturity and for all strikes, which, by [4], determines the distribution of the underlying asset at that maturity. This observation is crucial for the use of techniques from Optimal Transport and Skorokhod embedding, see [9, 2, 8, 10, 7, 1, 3, 5], among many others. On the other hand, Davis and Hobson [6] assume finitely many call options with possibly different maturities being traded in the market, and investigate consistency with absence of arbitrage.

In all these papers, the standing assumption is that there is no interest rate volatility, i.e. the interest rate is assumed to be deterministic or even constant (zero). Obviously the interest rates are never deterministic in real-world financial markets, however, to the best of our knowledge, so far there has been no paper taking this feature into account. It turns out that the presence of a stochastic interest rate has an important implication on the results that can be obtained.

In this paper we aim at addressing this issue by studying the relationship between quoted option prices and arbitrage opportunities in a fixed-income market. More precisely, we shall consider a market consisting of a zero-coupon bond with a given maturity as an underlying, and a set of call options with various maturities and various strikes written on it. In this framework we individuate necessary and sufficient conditions for the market prices to be consistent with the existence of a pricing measure, and relate these conditions to the concepts of weak and strong arbitrage. In contrast to the market studied in [6], that considers the case of a stock and call options written on it, our underlying asset is intrinsically related to interest rates. Assuming deterministic

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rates in our setup would thus imply that the underlying asset is also deterministic and the call options on it become trivial. Therefore, for any non-trivial set of call options there *cannot exist* a model with deterministic interest rate which is consistent with it. On the other hand, as we shall see in the rest of the paper, having stochastic interest rates allows for more freedom in constructing a model which is consistent with the given market prices for call options, and no inter-temporal conditions are needed.

Importantly, the bond price is a strictly positive process. Below we show that the positivity of the price of the underlying asset is closely related to an additional condition on the option prices in order to ensure no-arbitrage (see Condition (C3) and Remark 5). This condition is not present in [6] because the authors there allow for the underlying asset to have zero price with non-zero probability.

2 Notation and preliminary considerations

In a discrete-time setting $t = 0, 1, \cdots, T$, we consider a fixed-income market consisting of zero-coupon bonds and European call options written on the bond with maturity T. We do not assume an underlying probability space, but we rather take the model-independent approach and will draw conclusions only based on the observed market prices. As usual under this approach, we consider semi-static trading strategies: bonds are traded dynamically at any time before their maturity, while the call options are only traded at time zero and then kept till their maturity. The market is assumed to be perfectly liquid such that both bonds and calls can be traded in arbitrary amounts and without transaction costs. No assumption is made on the stochastic interest rate. Specifically, throughout the paper we work in the following setting.

Setting: On the bond with maturity T, a finite number of European call options is traded in the market, with maturities $T_1 < \cdots < T_{m-1}$, with $T_i \in \{1, \cdots, T-1\}$ for $i = 1, \cdots m-1$. We set $T_m = T$. For each $i = 1, \cdots, m$, we assume that the bond with maturity T_i is traded in the market. We shall refer to it in the sequel as T_i-bond, and denote by $(B_t(T_i))_{t \leq T_i}$ its (positive) price process. In particular, at time zero we observe the prices $B_0(T_i) > 0$ for all $i = 1, \cdots, m$. For a fixed maturity T_i, n_i call options are traded, with strikes $0 < K_1^i < \cdots < K_{n_i}^i$. Thus the pay-off of the (i,u)th call is given by $(B_{T_i}(T) - K_u^i)^+$, and we denote its quoted market price at time 0 by b_u^i.

Remark 1. In fixed-income markets, the most liquid options are actually not options on zero-coupon bonds, but rather options on the underlying interest rates such as caps (floors) and swaptions. An interest rate cap (floor) is an option providing protection against the discretely compounded interest rate exceeding (falling below) a certain fixed strike level K and it consists of a sequence of *caplets (floorlets)*, which are in fact put (call) options on the discretely compounded interest rate. By standard arguments in interest rate theory, it can be shown that the payoff of a caplet (floorlet) can be transformed into a payoff of a put (call) option on a zero-coupon bond, see e.g. [11], pages 439-440.

3 Definitions and statements of the main results

Definition 1 (Strong arbitrage)**.** *A* strong *(or* model-independent*) arbitrage opportunity is a self-financing semi-static portfolio in the bonds and available options, which has zero initial value, while all subsequent cash-flows are non-negative and at least at one date (strictly) positive.*

Definition 2 (Weak arbitrage)**.** *We say that there is a* weak arbitrage opportunity *if, given the null sets of the model, there is a self-financing semi-static portfolio, which has zero initial value, while all subsequent cash-flows are non-negative and the probability of a positive cash-flow is non-zero.*

Remark 2. Our definition of weak arbitrage differs from that used in [6]. Indeed, the existence of weak arbitrage in [6, Definition 2.3] in our terms translates as existence of weak but not strong arbitrage.

Remember that we work in the Setting described in Section 2.

Definition 3. *We say that the market prices admit a* compatible pricing model *if there is a probability space $(\Omega, \mathcal{F}, Q)$ supporting adapted processes $D = (D_t)_{t=0}^T$ and $B(T_i) = (B_t(T_i))_{t=0}^{T_i}, i = 1, \cdots, m$, such that:*
i) D is positive, with $D_0 = 1$;
ii) for $i = 1, \cdots, m$, the initial value of the process $B(T_i)$ is the T_i-bond price observed in the market, $B_0(T_i)$;
iii) $DB(T_i)$ is a Q-martingale for all $i = 1, \cdots, m$;
iv) for every $i = 1, \cdots, m$ and $u = 1, \ldots, n_i$, we have

$$E^Q[D_{T_i}(B_{T_i}(T) - K_u^i)^+] = b_u^i. \tag{1}$$

We denote by $\mathcal{M}^{mk}$ the collection of all such pricing models $(\Omega, \mathcal{F}, Q, D, B)$, and with an abuse of notation we will often write $Q \in \mathcal{M}^{mk}$.

Remark 3. Note that, if $\mathcal{M}^{mk} \neq \emptyset$ then there is no strong arbitrage. Indeed, let $Q \in \mathcal{M}^{mk}$, then, by the classical arguments used to prove the easy implication in the first FTAP, there is no self-financing semi-static portfolio which has zero initial value, and such that all subsequent cash-flows are Q-a.s. non-negative and the probability Q of a positive cash-flow is positive.

The aim of the paper is to provide necessary and sufficient conditions for the market prices to admit a compatible pricing model, and to relate it to absence of arbitrage (i.e. giving a version of the fundamental theorem of asset pricing in the present context). For this we need to introduce some notations.

Let us fix an $i \in \{1, \ldots, m\}$. Consider the following points consisting of pairs of strikes (scaled by a discount factor) and prices at time zero of the call options with maturity T_i written on the T-bond:

$$\left\{(0, B_0(T)), (B_0(T_i)K_u^i, b_u^i)_{u=1,\ldots,n_i}\right\}. \tag{2}$$

Note that the first point actually represents the call with strike 0, i.e. the bond itself, and its price at time zero $B_0(T)$. We connect these points by linear interpolation and consider the slopes between each two points, which are given as follows:

$$\begin{aligned} \alpha_1^i &:= \frac{b_1^i - B_0(T)}{B_0(T_i)K_1^i}, \\ \alpha_l^i &:= \frac{b_l^i - b_{l-1}^i}{B_0(T_i)(K_l^i - K_{l-1}^i)}, \quad \text{for} \quad l = 2, \ldots, n_i. \end{aligned} \tag{3}$$

We will show that the existence of a compatible pricing model is related to several conditions on the slopes α_l^i. The first condition requires that the curve obtained by linear interpolation of the points in (2) is non-increasing and convex, and that moreover the first slope is greater than or equal to -1.

Condition (C1). For all $i \in \{1, \ldots, m\}$, the slopes α_l^i, $l = 1, \ldots, n_i$, satisfy

$$-1 \leq \alpha_1^i \leq \alpha_2^i \leq \alpha_3^i \cdots \leq \alpha_{n_i}^i \leq 0.$$

Note that Condition (C1) implies that, if $\alpha^i_{l+1} = 0$ for some l, then $\alpha^i_u = 0$ and $b^i_u = b^i_l$ $\forall u \geq l+1$. In the next condition we ask for something more.

Condition (C2). For all $i \in \{1, \dots, m\}$, the following holds: if $\alpha^i_{l+1} = 0$ for some l, then $b^i_l = 0$.

Condition (C3). For all $i \in \{1, \dots, m\}$, the following holds: if $\alpha^i_2 = \alpha^i_1$, then $\alpha^i_2 = \alpha^i_1 = -1$.

We are now ready to state our main theorem, which we prove in Section 4.

Theorem 1. *We have the following implications:*

(I) Conditions (C1), (C2), (C3) hold if and only if the market prices admit a compatible pricing model, i.e. $\mathcal{M}^{mk} \neq \emptyset$*;*

(II) if there is no strong arbitrage, then Condition (C1) is satisfied;

(III) if there is no weak arbitrage, then Conditions (C2) and (C3) are satisfied.

Remark 4. Let us compare our results in Theorem 1 with those in [6] (remember that in the cited paper the authors consider a deterministic interest rate framework and calls are written on a non-negative financial asset S). The first thing to notice is that we do not need to consider inter-temporal relations between market prices. Our conditions (Ci) involve prices of options at every maturity separately. Indeed, our conditions for every maturity are comparable to the conditions in [6, Theorem 3.1] (single exercise time) rather than to those in [6, Theorem 4.2] (general multi-maturity case). The results of Davis and Hobson in the case of single exercise time then corresponds to our results in Theorem 1 where conditions (C3) is suppressed.

Corollary 1. *The following implications hold:*
no weak arbitrage $\Rightarrow$ $\mathcal{M}^{mk} \neq \emptyset$ $\Rightarrow$ *no strong arbitrage.*

Proof. Since clearly no weak arbitrage implies no strong arbitrage, then by Theorem 1 we have the first implication. That the second implication holds has been already observed in Remark 3. □

4 Auxiliary results and proof of Theorem 1

We first prove statement (II) of Theorem 1.

Lemma 1. *If there is no strong arbitrage, then Condition (C1) is satisfied.*

Proof. We are going to show that if Condition (C1) is not satisfied, there is model-independent arbitrage which we build explicitly in each case. Fix an i such that $i \in \{1, \dots, m\}$. This means that we are considering call prices on the T-bond with maturity T_i.

We begin by assuming that $\alpha^i_1 < -1$. Then at time 0 we buy the call option for b^i_1, sell the T-bond and buy K^i_1 units of the T_i-bond. This yields the initial profit of $-b^i_1 + B_0(T) - K^i_1 B_0(T_i)$, which is strictly positive by the above assumption. At exercise time T_i the value of the strategy is given by $(B_{T_i}(T) - K^i_1)^+ - B_{T_i}(T) + K^i_1 \geq 0$, and therefore there is an arbitrage opportunity (we can invest the initial profit of the above strategy in the T_i-bond).

Next we assume that $\alpha^i_1 > \alpha^i_2$ and construct an arbitrage opportunity stemming from it. We have

$$\frac{B_0(T) - b^i_1}{K^i_1} < \frac{b^i_1 - b^i_2}{K^i_2 - K^i_1}$$

and thus at time 0 we buy $\frac{1}{K_1^i}$ units of the T-bond, we sell $\frac{1}{K_1^i} + \frac{1}{K_2^i - K_1^i}$ units of the call with strike K_1^i and buy $\frac{1}{K_2^i - K_1^i}$ units of the call with strike K_2^i. This strategy is of a butterfly spread type, where buying the T-bond can be thought of as buying a call option on the T-bond with zero strike. The initial profit is given by $-\frac{1}{K_1^i} B_0(T) + \left(\frac{1}{K_1^i} + \frac{1}{K_2^i - K_1^i}\right) b_1^i - \frac{1}{K_2^i - K_1^i} b_2^i$, which is strictly positive by assumption above. At exercise time T_i the value of the strategy is given by

$$\frac{1}{K_1^i} B_{T_i}(T) - \left(\frac{1}{K_1^i} + \frac{1}{K_2^i - K_1^i}\right)(B_{T_i}(T) - K_1^i)^+ + \frac{1}{K_2^i - K_1^i}(B_{T_i}(T) - K_2^i)^+ \geq 0, \tag{4}$$

hence there is an arbitrage opportunity. Fix now an arbitrary $l \in \{2, \dots, n_i - 1\}$ and assume that $\alpha_l^i > \alpha_{l+1}^i$, which means

$$\frac{b_{l-1}^i - b_l^i}{K_l^i - K_{l-1}^i} < \frac{b_l^i - b_{l+1}^i}{K_{l+1}^i - K_l^i}.$$

The arbitrage strategy relies on a butterfly spread consisting of three calls with strikes K_{l-1}^i, K_l^i and K_{l+1}^i. More precisely, we buy $\frac{1}{K_l^i - K_{l-1}^i}$ units of the call with strike K_{l-1}^i, sell $\frac{1}{K_l^i - K_{l-1}^i} + \frac{1}{K_{l+1}^i - K_l^i}$ units of the call with strike K_l^i and buy $\frac{1}{K_{l+1}^i - K_l^i}$ units of the call with strike K_{l+1}^i. The initial profit is given by $-\frac{1}{K_l^i - K_{l-1}^i} b_{l-1}^i + \left(\frac{1}{K_l^i - K_{l-1}^i} + \frac{1}{K_{l+1}^i - K_l^i}\right) b_l^i - \frac{1}{K_{l+1}^i - K_l^i} b_{l+1}^i$, which is strictly positive by assumption. At exercise time T_i the value of this strategy is

$$\frac{1}{K_l^i - K_{l-1}^i}(B_{T_i}(T) - K_{l-1}^i)^+ - \left(\frac{1}{K_l^i - K_{l-1}^i} + \frac{1}{K_{l+1}^i - K_l^i}\right)(B_{T_i}(T) - K_l^i)^+ \\ + \frac{1}{K_{l+1}^i - K_l^i}(B_{T_i}(T) - K_{l+1}^i)^+ \geq 0$$

and therefore there is an arbitrage opportunity. The value is strictly positive on the set $\{K_{l-1}^i < B_{T_i}(T) < K_{l+1}^i\}$ and zero on its complement.

Finally, if $\alpha_l^i > 0$ for any $l \in \{1, \dots, n_i\}$, this means that $b_{l-1}^i < b_l^i$. In other words, the price of the call option with strike K_{l-1}^i is strictly smaller that the price of the call option with a bigger strike K_l^i, which creates an obvious arbitrage opportunity by buying at time 0 one call with strike K_{l-1}^i and selling one call with strike K_l^i. $\square$

We now prove statement (III) of Theorem 1.

Lemma 2. *If there is no weak arbitrage, then Conditions (C2) and (C3) are satisfied.*

Proof. First we show that no weak arbitrage implies Condition (C2). We do this by assuming that $\alpha_{l+1}^i = 0$ but $b_l^i > 0$, and show that in this case there is weak arbitrage. In this case we have $b_l^i = b_{l+1}^i > 0$. Then, in a model where $B_{T_i}(T) > K_l^i$ with positive probability, the strategy as in Definition 2 is obtained by buying at time zero a call option with strike K_l^i and selling a call option with strike K_{l+1}^i, which has value $-b_l^i + b_{l+1}^i = 0$. At T_i the value of the strategy is always positive and it is strictly positive on the set $\{B_{T_i}(T) > K_l^i\}$ On the other hand, in a model where $B_{T_i}(T) \leq K_l^i$ a.s., a strategy as in Definition 2 is obtained by selling at time zero a call option with strike K_l^i, and buying $b_l^i / B_0(T)$ T-bonds. This strategy has zero initial value, and is worth $b_l^i / B_0(T)$ at time T, since the call option expires worthless.

Now we show that no weak arbitrage implies Condition (C3). We consider the strategy described in the proof of Lemma 1, that consists in buying $\frac{1}{K_1^i}$ units of the T-bond, selling $\frac{1}{K_1^i}+\frac{1}{K_2^i-K_1^i}$ units of the call with strike K_1^i and buying $\frac{1}{K_2^i-K_1^i}$ units of the call with strike K_2^i. In case $\alpha_1^i=\alpha_2^i$, the initial value of such strategy is zero. On the other hand, its outcome at time T_i, given by (4), is strictly positive on the set $\{B_{T_i}(T)<K_2^i\}$ and zero on $\{B_{T_i}(T)\geq K_2^i\}$, because $B_{T_i}(T)>0$. Therefore, if $B_{T_i}(T)<K_2^i$ with positive probability, then there is a self-financing strategy as in Definition 2. On the other hand, if $B_{T_i}(T)\geq K_2^i$ a.s., then $B_{T_i}(T)\geq K_1^i$ a.s., and if $\alpha_1>-1$, there is a self-financing strategy as in Definition 2. Indeed, we can sell the call option for b_1^i, buy the T-bond and sell K_1^i units of the T_i-bond. This results in an initial value $p:=b_1^i-B_0(T)+K_1^iB_0(T_i)$, which is strictly positive because $\alpha_1>-1$, and is used to buy T_i-bonds. At time T_i, the value of such strategy is $-(B_{T_i}(T)-K_1^i)^++B_{T_i}(T)-K_1^i+p/B_0(T_i)=p/B_0(T_i)>0$ a.s.

□

The next lemma shows one implication of statement (I) in Theorem 1.

Lemma 3. *If $\mathcal{M}^{mk}\neq\emptyset$, then Conditions (C1), (C2) and (C3) hold.*

Proof. We assume $\mathcal{M}^{mk}\neq\emptyset$. Then Condition (C1) holds by Remark 3 and Lemma 1.

To show that Conditions (C2) (resp. (C3)) hold, we work by way of contradiction and assume this is not true. Then, we fix $Q\in\mathcal{M}^{mk}$ and use the same strategies built in the proof of Lemma 2 to obtain arbitrage strategies under Q, which is the desired contradiction. □

To complete the proof of our main theorem, we need to show sufficiency of Conditions (C1), (C2), (C3) for the existence of a compatible pricing model. We first show this in a simplified case, which will serve as building block in the proof of the general case.

Lemma 4. *Assume that the traded options on the T-bond all have the same maturity. If Conditions (C1), (C2) and (C3) hold, then $\mathcal{M}^{mk}\neq\emptyset$.*

Since we are in the simplified case of options with the same maturity, sat T_1, we can drop the indicator i on strikes, option prices, and slopes. This is what we do in the proof.

Proof. We begin by adding to the set of strikes a strike $K_{n+1}>K_n$ such that the points $\{(0,B_0(T)),(B_0(T_1)K_u,b_u)_{u=1,\dots,n},(B_0(T_1)K_{n+1},0)\}$, with the last slope given by $\alpha_{n+1}:=-\frac{b_n}{B_0(T_1)(K_{n+1}-K_n)}$, still satisfy Condition (C1). More precisely, thanks to (str) the point K_{n+1} can be chosen in such a way that $\alpha_n\leq\alpha_{n+1}\leq 0$. Indeed, if $\alpha_n<0$, then K_{n+1} can always be chosen large enough such that α_{n+1} satisfies the desired inequality. If $\alpha_n=0$, this implies by (C2) that $b_n=0$. Hence, $\alpha_{n+1}=0$ and the inequality is trivially satisfied. In this case adding the point K_{n+1} is not even needed and the martingale construction in the sequel can be done already with the last point being $(B_0(T_1)K_n,0)$.

The construction below depends on whether $\alpha_1<\alpha_2$ or $\alpha_1=\alpha_2$. Let us first assume that $\alpha_1<\alpha_2$. First we add one more point $K_0\in(0,K_1)$ between the points 0 and K_1. Later on we are going to choose K_0 in a suitable way. Next define $\Omega:=\{\omega_0,\dots,\omega_{n+1}\}$, $\mathcal{F}:=\mathcal{P}(\Omega)$ and Q such that $Q(\omega_i)=q_i$, $i=0,\dots,n+1$, with q_i's to be determined. For Q to be a well-defined probability measure, they have to satisfy $\sum_{i=0}^{n+1}q_i=1$ and $q_i\in[0,1]$, for every i. Set $D_{T_1}(\omega_i)=B_0(T_1)$ and define $k_i:=B_0(T_1)K_i$, for $i=0,\dots,n+1$. Now set for every i, $B_{T_1}(T)(\omega_i):=K_i$ and $D_T(\omega_i):=D_{T_1}(\omega_i)B_{T_1}(T)(\omega_i)=B_0(T_1)K_i$. The martingale condition on the discounted bond price process $(B_0(T), D_{T_1}B_{T_1}(T), D_T)$ thus translates then into the following equality that has to be satisfied by q_i's

$$\sum_{i=0}^{n+1} q_i k_i = B_0(T). \tag{5}$$

Obviously, $E^Q[D_T|\mathcal{F}_{T_1}] = D_{T_1}B_{T_1}(T)$. Moreover, matching the call prices in (1) we get the following n equations

$$q_{n+1}(k_{n+1} - k_n) = b_n \tag{6}$$

$$q_{n+1}(k_{n+1} - k_{n-1}) + q_n(k_n - k_{n-1}) = b_{n-1} \tag{7}$$

$$\ldots\ldots\ldots\ldots \tag{8}$$

$$\sum_{i=2}^{n+1} q_i(k_i - k_1) = b_1 \tag{9}$$

Thus, we have a system of $n+2$ equations with $n+2$ unknowns q_i, $i = 0, \ldots, n+1$. From the first of n equations for call prices we immediately get

$$q_{n+1} = \frac{b_n}{k_{n+1} - k_n} = -\alpha_{n+1}.$$

Furthermore, subtracting the first from the second equation and then the second from the third equation we get

$$q_n = \frac{b_{n-1} - b_n}{k_n - k_{n-1}} - q_{n+1} = \alpha_{n+1} - \alpha_n$$

$$q_{n-1} = \frac{b_{n-2} - b_{n-1}}{k_{n-1} - k_{n-2}} - (q_{n+1} + q_n) = \alpha_n - \alpha_{n-1}.$$

It is now easy to show by induction that for every $i = 2, \ldots, n$ we have

$$q_i = \frac{b_{i-1} - b_i}{k_i - k_{i-1}} - \sum_{j=i+1}^{n+1} q_i = \alpha_{i+1} - \alpha_i. \tag{10}$$

From equation (5) and using $q_0 = 1 - \sum_{i=1}^{n+1} q_i$ and equation (9), we deduce that

$$q_1 = -\left(\alpha_1 + \frac{k_0}{k_1}\right)\frac{k_1}{k_1 - k_0} + \alpha_2. \tag{11}$$

Finally, we find easily

$$q_0 = 1 - \sum_{i=1}^{n+1} q_i = 1 + \left(\alpha_1 + \frac{k_0}{k_1}\right)\frac{k_1}{k_1 - k_0}. \tag{12}$$

Recalling Condition (C1) and the assumption $\alpha_1 < \alpha_2$ above, we have $-1 \leq \alpha_1 < \alpha_2 \leq \alpha_3 \cdots \leq \alpha_{n+1} \leq 0$, which immediately implies $q_i \geq 0$ for $i = 2, \ldots, n+1$. Moreover, it implies that $\alpha_1 < 0$. To show now that $q_1 \geq 0$, we note that q_1 is a continuous function of k_0 and for $k_0 = 0$, $q_1 = \alpha_2 - \alpha_1 > 0$. Hence, we can choose $k_0 > 0$ small enough such that $q_1 \geq 0$. Similarly, for $k_0 > 0$ small enough, we have that $q_0 \geq 0$. Moreover, positivity of q's together with condition $\sum_{i=0}^{n+1} q_i = 1$ implies also $q_i \leq 1$, for all i and the proof for the case $\alpha_1 < \alpha_2$ is completed.

Now consider the case $\alpha_1 = \alpha_2$. By Condition (C3) we have $\alpha_1 = \alpha_2 = -1$. We shall proceed by constructing a model in a similar fashion as above, but such that $B_{T_1}(T)$ takes only the values $K_2, \dots, K_{n+1}$ which ensures $B_{T_1}(T) \geq K_2$. Define $\tilde{\Omega} := \{\tilde{\omega}_1, \dots, \tilde{\omega}_n\}$ and the probability measure $\tilde{Q}$ such that $\tilde{Q}(\tilde{\omega}_i) = \tilde{q}_i$, $i = 1, \dots, n$, with $\tilde{q}_i$'s to be determined. For $\tilde{Q}$ to be well-defined, they have to satisfy $\sum_{i=1}^n \tilde{q}_i = 1$ and $\tilde{q}_i \in [0,1]$, for every i. Set $D_{T_1}(\tilde{\omega}_i) = B_0(T_1)$ and define $k_i := B_0(T_1)K_i$, for $i = 1, \dots, n+1$. Now set for every i, $B_{T_1}(T)(\tilde{\omega}_i) := K_{i+1} \geq K_2$ and $D_T(\omega_i) := D_{T_1}(\omega_i)B_{T_1}(T)(\omega_i) = B_0(T_1)K_i$. The martingale condition on the discounted bond price process $(B_0(T), D_{T_1}B_{T_1}(T), D_T)$ translates then into the following equality that has to be satisfied by $\tilde{q}_i$'s

$$\sum_{i=1}^{n} \tilde{q}_i k_{i+1} = B_0(T). \tag{13}$$

and we obviously have $E^Q[D_T|\mathcal{F}_{T_1}] = D_{T_1}B_{T_1}(T)$. Moreover, matching the call prices in (1) we get the following n equations

$$\tilde{q}_n(k_{n+1} - k_n) = b_n \tag{14}$$

$$\tilde{q}_n(k_{n+1} - k_{n-1}) + \tilde{q}_{n-1}(k_n - k_{n-1}) = b_{n-1} \tag{15}$$

$$\cdots\cdots\cdots\cdots \tag{16}$$

$$\sum_{i=1}^{n} \tilde{q}_i(k_{i+1} - k_1) = b_1 \tag{17}$$

As above we have a system of $n+2$ equations, but with only n unknowns $\tilde{q}_i$, $i = 1, \dots, n$. From (14) we immediately get

$$\tilde{q}_n = \frac{b_n}{k_{n+1} - k_n} = -\alpha_{n+1}.$$

Furthermore, subtracting the first from the second equation and then the second from the third equation we get

$$\tilde{q}_{n-1} = \frac{b_{n-1} - b_n}{k_n - k_{n-1}} - \tilde{q}_n = \alpha_{n+1} - \alpha_n$$

$$\tilde{q}_{n-2} = \frac{b_{n-2} - b_{n-1}}{k_{n-1} - k_{n-2}} - (\tilde{q}_n + \tilde{q}_{n-1}) = \alpha_n - \alpha_{n-1}.$$

It is now easy to show by induction that for every $i = 1, \dots, n-3$ we have

$$\tilde{q}_i = \frac{b_{i-1} - b_i}{k_i - k_{i-1}} - \sum_{j=i+1}^{n} \tilde{q}_i = \alpha_{i+2} - \alpha_{i+1}. \tag{18}$$

Moreover, we have

$$\sum_{i=1}^{n} \tilde{q}_i = \sum_{i=2}^{n} (\alpha_{i+1} - \alpha_i) - \alpha_{n+1} = -\alpha_2 = 1.$$

Finally, using equation (17) and the fact that $\alpha_1 = -1$ we get

$$\sum_{i=1}^{n} \tilde{q}_i k_{i+1} = \sum_{i=1}^{n} \tilde{q}_i k_1 + b_1 = k_1 + b_1 = -\frac{B_0(T) - b_1}{\alpha_1} + b_1 = B_0(T) - b_1 + b_1 = B_0(T).$$

We complete the proof by noting that as above all $\tilde{q}$'s are positive by Condition (C1). □

Remark 5. We point out that the martingale construction in the proof above ensures $B_{T_1}(T)(\omega) > 0$, for all ω. This is one crucial difference with the construction proposed in [6], where the price of the underlying asset can be equal to zero with non-zero probability. In that paper, the cases $\alpha_1 < \alpha_2$ and $\alpha_1 = \alpha_2$ need not be treated separately. This distinction and the related Condition (C3) are linked precisely to the issue of strict positivity of the price of the underlying asset, as one can see from the above construction and from the proof of (1) in Lemma 2.

With the same reasoning we can deal with the case of deterministic interest rates and call options written on any strictly positive underlying.

Consider now the general case, where we take into account options traded on the T-bond with different maturities T_i, $i = 1, \ldots, m-1$. We prove the sufficiency of Conditions (C1), (C2), (C3) for the existence of a compatible pricing model, which shows the missing implication of statement (I) in Theorem 1.

Lemma 5. *If Conditions (C1), (C2) and (C3) hold, then* $\mathcal{M}^{mk} \neq \emptyset$.

Proof. We rely on the construction presented in the proof of Lemma 4 where the case of options with a common maturity was studied. The stepwise construction starts from the options with the shortest maturity and proceeds by enlarging the probability space as needed.

We begin by considering options with maturity T_1 and strikes K_u^1, $u = 1, \ldots, n_1$. The prices of the options are given by b_u^1, $u = 1, , \ldots, n_1$. Applying Lemma 4 for call options with maturity T_1, we obtain a probability space $(\Omega^1, \mathcal{F}, Q)$ and the random variables D_{T_1} and $B_{T_1}(T)$ such that $(B_0(T), D_{T_1} B_{T_1}(T))$ is a Q-martingale and for every $u = 1, \ldots, n_1$ the call price equation (1) from Definition 3 is satisfied. More precisely, assuming that $\alpha_1^1 < \alpha_2^1$ (the case $\alpha_1^1 = \alpha_2^1$ is treated similarly as in Lemma 4 using Condition (C3)), we define $\Omega^1 := \{\omega_0^1, \ldots, \omega_{n_1+1}^1\}$ and we set $D_{T_1}(\omega_u^1) = B_0(T_1)$ and define $k_u^1 := B_0(T_1) K_u^1$, for $u = 0, \ldots, n_1 + 1$, where we add the strikes K_0^1 and $K_{n_1+1}^1$ following Lemma 4. For every u, $B_{T_1}(T)(\omega_u^1) = K_u^1$. The probabilities $q_u^1 = Q(\omega_u^1)$ are defined as in Lemma 4. We thus have, again by Lemma 4, that $(B_0(T), D_{T_1} B_{T_1}(T))$ is indeed a Q-martingale and the call prices (1) are matched. Moreover, we define the σ-algebra $\mathcal{F}_{T_1}$ by $\mathcal{F}_{T_1} := \sigma(D_{T_1}, B_{T_1}(T)) = \sigma(B_{T_1}(T))$.

In the next step we have to construct the random variables $B_{T_2}(T)$ and D_{T_2} such that

$$E^Q[D_{T_2} B_{T_2}(T) | \mathcal{F}_{T_1}] = D_{T_1} B_{T_1}(T) = B_0(T_1) B_{T_1}(T) \tag{19}$$

and the call price equation (1) is satisfied for options with maturity T_2. We will proceed again by following Lemma 4 applied to the options with maturity T_2 and strikes K_u^2, for $u = 1, \ldots, n_2$. We assume again that $\alpha_1^2 < \alpha_2^2$ (the case $\alpha_1^2 = \alpha_2^2$ is treated similarly as in Lemma 4 using Condition (C3)) and we again add the strikes K_0^2 and $K_{n_2+1}^2$ as described in this lemma. We define $k_v^2 := B_0(T_2) K_v^2$, for $v = 0, \ldots, n_2 + 1$.

Now we enlarge the probability space by introducing the product space $\Omega^1 \times \Omega^2 = \{(\omega_u^1, \omega_v^2) : u = 0, \ldots, n_1+1, v = 0, \ldots, n_2+1\}$. We define for every $(\omega_u^1, \omega_v^2) \in \Omega^1 \times \Omega^2$

$$B_{T_2}(T)(\omega_u^1, \omega_v^2) = K_v^2$$

and set

$$D_{T_2}(\omega_u^1, \omega_v^2) = B_0(T_2) \frac{B_0(T_1)}{B_0(T)} B_{T_1}(T)(\omega_u^1, \omega_v^2).$$

We now have to determine the probabilities $q_v^2 \in [0, 1]$ such that equation (19) is satisfied and for every $\bar{v} \in \{1, \ldots, n_2\}$ the call price equation (1) is satisfied for the strike $K_{\bar{v}}^2$ and price $b_{\bar{v}}^2$. More precisely, equation (19) is equivalent to

$$\sum_{v=0}^{n_2+1} q_v^2 k_v^2 = B_0(T)$$

and the call price equations take the same form as in the proof of Lemma 4. Thus, Condition (C1) on the slopes α_v^2 ensures that probabilities $q_v^2 \in [0,1]$ defined as in Lemma 4 for options with maturity T_2 are indeed well-defined.

Proceeding in the same manner, we construct finally a sequence of random variables $B_{T_i}(T)$ and D_{T_i}, for $i < m$, and set in addition $D_T = D_{T_m} = D_{T_{m-1}} B_{T_{m-1}}(T_m)$ such that $(B_0(T), D_{T_1} B_{T_1}(T), \ldots, D_T)$ is a Q-martingale and all call price equations (1) are satisfied. The bond prices $B_{T_k}(T_i)$, for $T_k < T_i$ and $T_i < T = T_m$, are defined via the relationship

$$B_{T_k}(T_i) = \frac{1}{D_{T_k}} E^Q[D_{T_i} | \mathcal{F}_{T_k}].$$

□

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