# On the Lower Arbitrage Bound of American Contingent Claims*

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**Abstract**

We prove that in a discrete-time market model the lower arbitrage bound of an American contingent claim is itself an arbitrage-free price if and only if it corresponds to the price of the claim optimally exercised under some equivalent martingale measure.

**Keywords**: American contingent claim, arbitrage-free price, Snell envelope
**MSC2010**: 91B24,91G99.

## 1 Introduction

When teaching an introductory class in mathematical finance, a very convenient and natural approach is to develop the arbitrage pricing theory first in the case of a complete market, i.e., when there exists just one equivalent martingale measure, and then to extend this to the incomplete case, i.e., to the case of more than one equivalent martingale measure. The advantage of this approach is that in general whatever operation is carried out under the unique equivalent martingale measure in order to price a contingent claim in the complete case, is done under any equivalent martingale measure in the incomplete one. For instance the pricing of a (discounted) European contingent claim with payoff $Y$ in a complete arbitrage-free market amounts to simply taking the expectation of $Y$ under the unique equivalent martingale measure $\mathbb{Q}$, thus yielding the unique arbitrage-free price $\pi = E^{\mathbb{Q}}[Y]$. Respectively, in the incomplete case we have a set of arbitrage-free prices which is the set of expectations $E^{\mathbb{Q}}[Y]$ of $Y$ under all equivalent martingale measures $\mathbb{Q}$. However, when introducing American contingent claims and developing the corresponding arbitrage pricing theory, the extension from complete to incomplete markets causes unexpected problems. We recall that an American contingent claim is a contract which obliges the seller to pay a certain amount $H_\sigma \geq 0$ if the buyer of that claim decides to exercise the claim at (the stopping) time $\sigma$. Hence such an American contingent claim $H$ is determined by the process $H = (H_t)_t$ of its possible (discounted) payoffs $H_t$ at any trading time $t$. Considering the complete market case with a unique equivalent

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martingale measure $\mathbb{Q}$, the pricing of the (discounted) American contingent claim boils down to an optimal stopping problem. Indeed, knowing that there exists an optimal exercise time $\tau$, i.e., a stopping time $\tau$ maximizing the expected payoff of $H$ under $\mathbb{Q}$ over all possible payoffs,

$$E^{\mathbb{Q}}[H_\tau] = \sup\{E^{\mathbb{Q}}[H_\sigma] \mid \sigma \text{ is an exercise time }\}, \tag{1.1}$$

the unique arbitrage-free price of $H$ is $\pi = E^{\mathbb{Q}}[H_\tau]$. Notice that this price is in particular the unique fair price in the sense of neither being too expensive from the buyer's point of view, since she may exercise it optimally at $\tau$, nor being too cheap from the seller's point of view, as there is no exercise strategy $\sigma$ such that the value of the exercised claim $H_\sigma$ exceeds $\pi$. When pricing an American contingent claim in an incomplete market model, one would, on the one hand, expect the set of arbitrage-free prices to correspond to the set of prices obtained from the solutions to the optimal stopping problem (1.1) under any equivalent martingale measure. On the other hand, any arbitrage-free price should also be fair in the above sense. However, so far, this was not quite clear.

Following e.g. [2] we will define the arbitrage-free prices of a (discounted) American contingent claim $H$ in an incomplete market as the fair prices in the above sense; see Definition 2.5. It is then easily verified that the prices corresponding to the solutions to (1.1) under any equivalent martingale measure are arbitrage-free. However, the converse, that is the fact that every arbitrage-free price of an American contingent claim originates from the solution to (1.1) under some equivalent martingale measure, has not been clear so far. To be more precise, the problem here is the lower arbitrage bound, i.e., the infimum over all arbitrage-free prices, which in case of an American contingent claim may or may not be itself an arbitrage-free price. In case the lower arbitrage bound $\underline{\pi}(H)$ of $H$ is an arbitrage-free price, it was an open question whether there exists a minimal equivalent martingale measure in the sense that the solution to (1.1) under that measure yields the price $\underline{\pi}(H)$. In this paper, we prove that this is indeed the case, and we also give characterizations of this situation in terms of replicability properties of $H$ (Theorem 3.7). Thus we conclude that the two approaches to arbitrage pricing, i.e., extending the very natural optimal stopping approach from the complete to the incomplete case and the presented notion of a fair price, are indeed consistent. This is formulated in our main result which is stated in Theorem 2.8. In his doctoral thesis [9], Trevino Aguilar studies a closely related problem in a continuous-time framework, and many techniques we apply in our proofs are adopted from this work; see Remark 3.8.

The remainder of the paper is organized as follows: In Section 2 we introduce the market model, give a short overview over the arbitrage pricing theory as regards American contingent claims and state our main result in Theorem 2.8. The proof of Theorem 2.8 is then carried out in Section 3, at the end of which we give a more precise characterization of the case when the lower arbitrage bound is itself an arbitrage-free price (Theorem 3.7). Finally, in Section 4 we provide an example illustrating our main results.

We assume that the reader is familiar with standard multi-period discrete-time arbitrage theory such as outlined in Föllmer and Schied [2]. The book [2] is our main reference, and our setup and notation will to a major extent be adopted from there. As regards the arbitrage pricing theory of American contingent claims and the related theory of Snell envelopes, we also refer the reader to [1, 3, 4, 6, 8].

## 2 The Main Result

Throughout this paper we consider a discrete-time market model in which $d$ assets are priced at times $t = 0, \dots, T$ with $T \in \mathbb{N}$. The information available in the market is modeled by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, \mathbb{P})$, where $\mathcal{F}_t$ is the class of all events observable up to time $t \in \{0, \dots, T\}$, with

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \quad \text{and} \quad \mathcal{F}_T = \mathcal{F}.$$

All equalities and inequalities between random variables are understood in the $\mathbb{P}$-almost sure sense.

Following standard arbitrage theory, we assume the existence of a strictly positive asset which is used as numéraire for discounting. We indicate by $S^i = (S^i_t)_{t=0,\dots,T}$, $i = 1, \dots, d$, the discounted price process of asset $i$, which is assumed to be non-negative and adapted to the filtration $(\mathcal{F}_t)_{t=0,\dots,T}$. A $d$-dimensional process $\xi = (\xi^1, \dots, \xi^d) = (\xi^1_t, \dots, \xi^d_t)_{t=1,\dots,T}$ is a trading strategy if it is predictable in the sense that $\xi^i_t$ is $\mathcal{F}_{t-1}$-measurable for all $t = 1, \dots, T$ and $i = 1, \dots, d$. Here $\xi^i_t$ represents the quantity of the asset $i$ kept in the portfolio between time $t-1$ and time $t$. A trading strategy $\xi$ is called self-financing if at the trading times $t = 1, \dots T-1$ the portfolio is rebalanced between the assets without adding or withdrawing money, that is, if

$$\sum_{i=1}^{d} \xi^i_t S^i_t = \sum_{i=1}^{d} \xi^i_{t+1} S^i_t \quad \text{for } t = 1, \dots T-1.$$

The (discounted) value process corresponding to a trading strategy $\xi$ is

$$V^\xi = (V^\xi_t)_{t=0,\dots,T}, \quad \text{where } V^\xi_0 = \sum_{i=1}^{d} \xi^i_1 S^i_0 \text{ and } V^\xi_t = \sum_{i=1}^{d} \xi^i_t S^i_t,\ t = 1, \dots, T.$$

**Definition 2.1.** *A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is called an equivalent martingale measure if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and $S = (S^1, \dots, S^d)$ is a (d-dimensional) martingale under $\mathbb{Q}$. The set of all equivalent martingale measures is denoted by $\mathcal{M}$.*

We make the following assumption, which is notably equivalent to the No-Arbitrage condition for the market $S$.

**Assumption 2.2.** *We assume that $\mathcal{M} \neq \emptyset$.*

In our study of pricing American contingent claims, the simpler form of a European contingent claim will play an important role.

**Definition 2.3.** *A (discounted) European contingent claim is a non-negative random variable $Y$ on $(\Omega, \mathcal{F}, \mathbb{P})$.*

We indicate by $\Pi(Y)$ the set of arbitrage-free prices of $Y$, that is,

$$\Pi(Y) = \{E^{\mathbb{Q}}[Y] \mid \mathbb{Q} \in \mathcal{M} \text{ and } E^{\mathbb{Q}}[Y] < \infty\}. \tag{2.1}$$

It is well-know that either $\Pi(Y)$ is an open interval or $\Pi(Y)$ is a singleton, and that the latter case is equivalent to $Y$ being replicable, which means that there exists a self-financing trading strategy $\xi$ such that $V^\xi_T = Y$; see, e.g., [2, Theorem 5.33] and [5].

**Definition 2.4.** *A (discounted) American contingent claim is a non-negative adapted process* $H = (H_t)_{t=0,\dots,T}$ *on the filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, \mathbb{P})$.

Let $\mathcal{T}$ denote the set of stopping times $\tau : \Omega \to \{0, \dots, T\}$. For each time $\tau \in \mathcal{T}$ the random variable $H_\tau$ is interpreted as the discounted payoff obtained by exercising the American contingent claim $H$ at time $\tau$. Note that $H_\tau$ can be considered as the discounted payoff of a European contingent claim. Therefore, the set of arbitrage-free prices of $H_\tau$ is given by

$$\Pi(H_\tau) = \{E^{\mathbb{Q}}[H_\tau] \mid \mathbb{Q} \in \mathcal{M} \text{ and } E^{\mathbb{Q}}[H_\tau] < \infty\}.$$

We define the set of arbitrage-free prices of an American contingent claim as in [2, Definition 6.31] reflecting the asymmetric connotation of such a contract: the seller must hedge against all possible exercise times, while the buyer only needs to find one favorable exercise strategy.

**Definition 2.5.** *A real number* $\pi$ *is an arbitrage-free price of the discounted American contingent claim* $H$ *if the following two conditions are satisfied:*

(i) *There exists some* $\tau \in \mathcal{T}$ *and* $\pi' \in \Pi(H_\tau)$ *such that* $\pi \leq \pi'$.

(ii) *There is no* $\tau \in \mathcal{T}$ *such that* $\pi < \pi'$ *for all* $\pi' \in \Pi(H_\tau)$.

The interpretation of the two requirements in Definition 2.5 is clear. The first one makes the proposed price $\pi$ not too high from the buyer's point of view, in the sense that there exists some exercise strategy $\tau \in \mathcal{T}$ and a $\mathbb{Q} \in \mathcal{M}$ such that $\pi \leq E^{\mathbb{Q}}[H_\tau]$. The second requirement accounts for the point of view of the seller, ruling out exercise strategies $\tau \in \mathcal{T}$ such that $\pi < E^{\mathbb{Q}}[H_\tau]$ for all $\mathbb{Q} \in \mathcal{M}$.

We denote by $\Pi(H)$ the set of all arbitrage-free prices of an American contingent claim $H$. Notice that if $H$ corresponds to a European contingent claim with maturity $T$, i.e.

$$H_t = 0 \text{ for all } t = 0, \dots, T-1, \text{ and } H_T = Y \tag{2.2}$$

for some non-negative $\mathcal{F}$-measurable random variable $Y$, we have that $\Pi(H) = \Pi(Y)$, where $\Pi(Y)$ is given in (2.1). Hence the pricing rules are consistent.

For the remainder of the paper we consider a fixed American contingent claim $H$ such that

$$H_t \in L^1(\Omega, \mathcal{F}, \mathbb{Q}) \quad \text{for all} \quad t = 0, \dots, T \quad \text{and} \quad \mathbb{Q} \in \mathcal{M}. \tag{2.3}$$

**Proposition 2.6.** *(see [2, Theorem 6.33]) Under condition* (2.3), *the set* $\Pi(H)$ *of all arbitrage-free prices for* $H$ *is a real interval with endpoints*

$$\underline{\pi}(H) = \inf_{\mathbb{Q} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} E^{\mathbb{Q}}[H_\tau] = \sup_{\tau \in \mathcal{T}} \inf_{\mathbb{Q} \in \mathcal{M}} E^{\mathbb{Q}}[H_\tau]$$

*and*

$$\overline{\pi}(H) = \sup_{\mathbb{Q} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} E^{\mathbb{Q}}[H_\tau] = \sup_{\tau \in \mathcal{T}} \sup_{\mathbb{Q} \in \mathcal{M}} E^{\mathbb{Q}}[H_\tau].$$

*Moreover,* $\Pi(H)$ *either consists of one single point or does not contain its upper endpoint* $\overline{\pi}(H)$.

As for European contingent claims, $\overline{\pi}(H) \in \Pi(H)$ implies that $\Pi(H) = \{\overline{\pi}(H)\}$, which is equivalent to $H$ being attainable in the sense that there exists a superhedging strategy $\xi$ (for the seller) such that

$$V_0^\xi = \overline{\pi}(H),\ V_t^\xi \geq H_t \text{ for all } t = 1, \dots, T, \text{ and } V_\tau^\xi = H_\tau \text{ for a } \tau \in \mathcal{T},$$

see [2, Theorem 6.36]. This means that, if the buyer exercises the claim at time $\tau$, then she meets the value of the seller's hedging portfolio. This replicability concept again corresponds to the replicability of a European contingent claim if $H$ is of type (2.2). However, in contrast to the pricing of a European contingent claim, in case of a non-replicable American contingent claim both cases

$$\underline{\pi}(H) \in \Pi(H) \quad \text{and} \quad \underline{\pi}(H) \notin \Pi(H)$$

can occur, see Example 4.1.

**Definition 2.7.** *A stopping time $\tau \in \mathcal{T}$ is an optimal stopping time for $H$ under $\mathbb{Q} \in \mathcal{M}$ if*

$$E^{\mathbb{Q}}[H_\tau] = \sup_{\sigma \in \mathcal{T}} E^{\mathbb{Q}}[H_\sigma]. \tag{2.4}$$

It is well-known that set of optimal stopping times for $H$ under any $\mathbb{Q} \in \mathcal{M}$ is non-empty; see [2, Theorem 6.20]. Moreover, it is easily verified that, if $\tau$ is an optimal stopping time under some $\mathbb{Q} \in \mathcal{M}$, then $E^{\mathbb{Q}}[H_\tau] \in \Pi(H)$. Note also that the set

$$\mathcal{P} := \{E^{\mathbb{Q}}[H_\tau] \mid \mathbb{Q} \in \mathcal{M} \text{ and } \tau \in \mathcal{T} \text{ is optimal under } \mathbb{Q}\ \}$$

is an interval. This is due to the fact that for any two equivalent martingale measures $\mathbb{P}_0, \mathbb{P}_1 \in \mathcal{M}$ and for all $\alpha \in [0,1]$ we have that $\alpha\mathbb{P}_0 + (1-\alpha)\mathbb{P}_1 \in \mathcal{M}$, and the function

$$f : [0,1] \to \mathbb{R}, \quad \alpha \mapsto \sup_{\sigma \in \mathcal{T}} \left(\alpha E^{\mathbb{P}_0}[H_\sigma] + (1-\alpha) E^{\mathbb{P}_1}[H_\sigma]\right)$$

is continuous. Of course the interval bounds of $\mathcal{P}$ are $\underline{\pi}(H)$ and $\overline{\pi}(H)$. So according to Proposition 2.6, if $\underline{\pi}(H) \notin \Pi(H)$, then $\mathcal{P} = \Pi(H)$. However, it has been an open question whether $\mathcal{P} = \Pi(H)$ also in case $\underline{\pi}(H) \in \Pi(H)$. If $\underline{\pi}(H) \in \Pi(H)$, the problem is whether there exists an equivalent martingale measure $\mathbb{Q} \in \mathcal{M}$ and an optimal stopping time $\tau$ under $\mathbb{Q}$ such that $E^{\mathbb{Q}}[H_\tau] = \underline{\pi}(H)$, that is,

$$\sup_{\sigma \in \mathcal{T}} E^{\mathbb{Q}}[H_\sigma] = \inf_{P \in \mathcal{M}} \sup_{\sigma \in \mathcal{T}} E^P[H_\sigma]. \tag{2.5}$$

In Theorem 2.8, which is our main result, we show that this is indeed the case, that is, there is a one-to-one correspondence between the set $\Pi(H)$ and the solutions to problem (2.4).

**Theorem 2.8.** *The set $\Pi(H)$ of all arbitrage-free prices for $H$ coincides with the set $\mathcal{P}$ of evaluations at optimal times:*

$$\mathcal{P} = \Pi(H).$$

The proof of Theorem 2.8 needs some preparation which will be carried out in Section 3, at the end of which, in Theorem 3.7, we state and prove that $\underline{\pi}(H) \in \Pi(H)$ if and only if there exists $\mathbb{Q} \in \mathcal{M}$ satisfying (2.5). Moreover, we also give a detailed characterization of this situation in terms of the replicability of a European contingent claim corresponding to exercising $H$ at a specific stopping time.

## 3 Discussion and Proof of Theorem 2.8

In what follows we introduce the basic tools needed for our analysis of the lower arbitrage bound.

**Definition 3.1.** *For $\mathbb{Q} \in \mathcal{M}$, the Snell envelope $U^{\mathbb{Q}} = (U^{\mathbb{Q}}_t)_{t=0,\dots,T}$ of the American contingent claim $H$ with respect to the measure $\mathbb{Q}$ is defined by*

$$U^{\mathbb{Q}}_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}, \tau \geq t} E^{\mathbb{Q}}[H_\tau \mid \mathcal{F}_t], \quad t = 0, \dots, T.$$

The process $U^{\mathbb{Q}}$ is the smallest $\mathbb{Q}$-supermartingale dominating $H$. In particular, the value at zero of the Snell envelope with respect to a measure $\mathbb{Q} \in \mathcal{M}$ is the value obtained by optimally exercising the American contingent claim under that measure:

$$U^{\mathbb{Q}}_0 = \sup_{\sigma \in \mathcal{T}} E^{\mathbb{Q}}[H_\sigma].$$

It is known that $\tau \in \mathcal{T}$ is an optimal stopping time for $H$ under $\mathbb{Q}$ if and only if $H_\tau = U^{\mathbb{Q}}_\tau$ and the stopped process $(U^{\mathbb{Q}})^\tau := (U^{\mathbb{Q}}_{\tau \wedge t})_{t=0,\dots,T}$ is a $\mathbb{Q}$-martingale; see [2, Proposition 6.22]. We indicate with $\tau^{\mathbb{Q}}$ the minimal optimal stopping time for $H$ under $\mathbb{Q} \in \mathcal{M}$, which is equal to the first time where the Snell envelope of $H$ w.r. to $\mathbb{Q}$ equals the value of the American contingent claim, that is,

$$\tau^{\mathbb{Q}} = \inf\{t \geq 0 \mid U^{\mathbb{Q}}_t = H_t\}.$$

Throughout the paper a predominant role is played by the following stopping time:

$$\hat{\tau} := \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{M}} \tau^{\mathbb{Q}},$$

the importance of which will become clear in Theorem 3.7. We will amongst others show that the lower arbitrage bound $\underline{\pi}(H)$ of $H$ is itself an arbitrage-free price if and only if the European contingent claim $H_{\hat{\tau}}$ can be replicated. Let us first verify that $\hat{\tau}$ is indeed a stopping time.

**Lemma 3.2.** *The set $\{\tau^{\mathbb{Q}} \mid \mathbb{Q} \in \mathcal{M}\}$ is downward directed. Hence, $\hat{\tau}$ is a stopping time. In particular, there exists a sequence $(\mathbb{Q}_k)_{k \in \mathbb{N}} \in \mathcal{M}$ such that $\{\tau^{\mathbb{Q}_k} = \hat{\tau}\} \nearrow \Omega$ for $k \to \infty$.*

*Proof.* For the first part we repeat the argument in the proof of [9, Theorem 5.6] to show that the set $\{\tau^{\mathbb{Q}} \mid \mathbb{Q} \in \mathcal{M}\}$ is downward directed. Indeed, let $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{M}$, $B := \{\tau^{\mathbb{Q}_1} \geq \tau^{\mathbb{Q}_2}\}$, and let $\widetilde{\mathbb{Q}}$ be the measure obtained by pasting $\mathbb{Q}_1$ and $\mathbb{Q}_2$ in the stopping time $\sigma := \left(\tau^{\mathbb{Q}_1} \wedge \tau^{\mathbb{Q}_2}\right) 1_B + T 1_{B^c}$, that is

$$\widetilde{\mathbb{Q}}(A) := E^{\mathbb{Q}_1}\left[E^{\mathbb{Q}_2}\left[1_A \mid \mathcal{F}_\sigma\right]\right], \quad A \in \mathcal{F}.$$

It is then verified that

$$U^{\widetilde{\mathbb{Q}}}_{\tau^{\mathbb{Q}_1} \wedge \tau^{\mathbb{Q}_2}} = U^{\mathbb{Q}_2}_{\tau^{\mathbb{Q}_2}} 1_B + U^{\mathbb{Q}_1}_{\tau^{\mathbb{Q}_1}} 1_{B^c} = H_{\tau^{\mathbb{Q}_1} \wedge \tau^{\mathbb{Q}_2}},$$

hence $\tau^{\widetilde{\mathbb{Q}}} \leq \tau^{\mathbb{Q}_1} \wedge \tau^{\mathbb{Q}_2}$, so $\{\tau^{\mathbb{Q}} \mid \mathbb{Q} \in \mathcal{M}\}$ is downward directed. This implies that there is a sequence $(\mathbb{Q}_k)_{k \in \mathbb{N}} \subset \mathcal{M}$ such that $\tau^{\mathbb{Q}_k} \searrow \hat{\tau}$. ¿From that it follows that $\hat{\tau} = \operatorname{ess\,inf}\{\tau^{\mathbb{Q}_k} \mid k \in \mathbb{N}\}$ is a stopping time. Moreover, as time is discrete and by the monotonicity of the sequence $(\tau^{\mathbb{Q}_k})_{k \in \mathbb{N}}$, we deduce that $\{\tau^{\mathbb{Q}_k} = \hat{\tau}\} \nearrow \Omega$ for $k \to \infty$. $\square$

**Definition 3.3.** *The lower Snell envelope* $U^{\downarrow} = (U_t^{\downarrow})_{t=0,\dots,T}$ *of the process* $H$ *(w.r.t. to* $\mathcal{M}$*) is defined by*

$$U_t^{\downarrow} = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}} U_t^{\mathbb{Q}} = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}} \operatorname*{ess\,sup}_{\tau\in\mathcal{T},\tau\geq t} E^{\mathbb{Q}}[H_\tau \mid \mathcal{F}_t], \quad t = 0,\dots,T.$$

*In particular,* $U_0^{\downarrow} = \underline{\pi}(H)$.

Notice that, according to Lemma 3.2, for almost all $\omega \in \Omega$ we have $\hat{\tau}(\omega) = \tau^{P_\omega}(\omega)$ for some $P_\omega \in \mathcal{M}$. Hence we obtain that for almost all $\omega$

$$H_{\hat{\tau}}(\omega) = H_{\tau^{P_\omega}}(\omega) = U_{\tau^{P_\omega}}^{P_\omega}(\omega) \geq U_{\tau^{P_\omega}}^{\downarrow}(\omega) = U_{\hat{\tau}}^{\downarrow}(\omega) \geq H_{\hat{\tau}}(\omega).$$

Consequently,

$$U_{\hat{\tau}}^{\downarrow} = H_{\hat{\tau}}. \tag{3.1}$$

The stopping time $\hat{\tau}$ is intensively studied in [9], which treats lower (and upper) Snell envelopes in general. Proposition 3.4 extends a result obtained in [9].

**Proposition 3.4.** *The lower Snell envelope* $U^{\downarrow}$ *satisfies the following properties:*

(i) $(U^{\downarrow})^{\hat{\tau}}$ *is a* $\mathcal{M}$*-submartingale, i.e., a submartingale under each* $\mathbb{Q} \in \mathcal{M}$.

(ii) *If* $H_{\hat{\tau}}$ *is replicable at price* $\underline{\pi}(H)$*, then* $(U^{\downarrow})^{\hat{\tau}}$ *is a* $\mathcal{M}$*-martingale.*

*Proof.* (i): Note that the set $\{U_t^{\mathbb{Q}} \mid \mathbb{Q} \in \mathcal{M}\}$ is downward directed for every $t \in \{0,\dots,T\}$. Indeed, let $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{M}$ and, as done in [2, Lemma 6.50], define the set $B := \{U_t^{\mathbb{Q}_1} > U_t^{\mathbb{Q}_2}\} \in \mathcal{F}_t$ and the probability measure $\widetilde{\mathbb{Q}}$ obtained by pasting $\mathbb{Q}_1$ and $\mathbb{Q}_2$ in the stopping time $\sigma := t1_B + T1_{B^c}$, that is,

$$\widetilde{\mathbb{Q}}(A) = E^{\mathbb{Q}_1}\left[E^{\mathbb{Q}_2}[1_{A\cap B} \mid \mathcal{F}_t] + 1_{A\cap B^c}\right], \quad A \in \mathcal{F}.$$

It is then verified that

$$U_t^{\widetilde{\mathbb{Q}}} = U_t^{\mathbb{Q}_2}1_B + U_t^{\mathbb{Q}_1}1_{B^c} = U_t^{\mathbb{Q}_1} \wedge U_t^{\mathbb{Q}_2}.$$

Therefore there is a sequence $(\mathbb{Q}_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ such that

$$U_t^{\mathbb{Q}_k} \searrow U_t^{\downarrow}.$$

Now fix $\mathbb{P}^* \in \mathcal{M}$ and notice that we may assume $\mathbb{Q}_k|_{\mathcal{F}_t} = \mathbb{P}^*|_{\mathcal{F}_t}$ for all $k$. Indeed, by pasting $\mathbb{P}^*$ with $\mathbb{Q}_k$ in $t$ we obtain a measure $\mathbb{Q}_k^* \in \mathcal{M}$ given by

$$\mathbb{Q}_k^*(A) = E^{\mathbb{P}^*}[E^{\mathbb{Q}_k}[1_A \mid \mathcal{F}_t]], \quad A \in \mathcal{F},$$

which coincides with $\mathbb{P}^*$ on $\mathcal{F}_t$ and is such that $U_t^{\mathbb{Q}_k^*} = U_t^{\mathbb{Q}_k}$. Now for every $t \in \{1,\dots,T\}$,

$$E^{\mathbb{P}^*}[U_{\hat{\tau}\wedge t}^{\downarrow} \mid \mathcal{F}_{t-1}] = U_{\hat{\tau}}^{\downarrow}1_{\{\hat{\tau}\leq t-1\}} + E^{\mathbb{P}^*}[U_t^{\downarrow} \mid \mathcal{F}_{t-1}]1_{\{\hat{\tau}\geq t\}}$$

and

$$\begin{aligned} E^{\mathbb{P}^*}\left[U_t^{\downarrow} \mid \mathcal{F}_{t-1}\right]1_{\{\hat{\tau}\geq t\}} &= E^{\mathbb{P}^*}\left[\lim_{k\to\infty} U_t^{\mathbb{Q}_k} \mid \mathcal{F}_{t-1}\right]1_{\{\hat{\tau}\geq t\}} \\ &= \lim_{k\to\infty} E^{\mathbb{P}^*}\left[U_t^{\mathbb{Q}_k} \mid \mathcal{F}_{t-1}\right]1_{\{\hat{\tau}\geq t\}} \\ &= \lim_{k\to\infty} E^{\mathbb{Q}_k}\left[U_{\tau^{\mathbb{Q}_k}\wedge t}^{\mathbb{Q}_k} \mid \mathcal{F}_{t-1}\right]1_{\{\hat{\tau}\geq t\}} \\ &= \lim_{k\to\infty} U_{\tau^{\mathbb{Q}_k}\wedge(t-1)}^{\mathbb{Q}_k}1_{\{\hat{\tau}\geq t\}} \\ &= \lim_{k\to\infty} U_{t-1}^{\mathbb{Q}_k}1_{\{\hat{\tau}\geq t\}} \quad \geq \quad U_{t-1}^{\downarrow}1_{\{\hat{\tau}\geq t\}}, \end{aligned}$$

where we use the dominated convergence theorem in the second equality since $0 \leq U_t^{\mathbb{Q}_k} \leq U_t^{\mathbb{Q}_1} \leq \sum_{t=0}^{T} H_t$ , and the facts that $\mathbb{Q}_k|_{\mathcal{F}_t} = \mathbb{P}^*|_{\mathcal{F}_t}$, $\hat{\tau} \leq \tau^{\mathbb{Q}}$, and $(U^{\mathbb{Q}_k})^{\tau^{\mathbb{Q}_k}}$ is a $\mathbb{Q}_k$-martingale for the rest. As $\mathbb{P}^* \in \mathcal{M}$ was arbitrary, (i) is proved.

(ii): Let $H_{\hat{\tau}}$ be replicable at price $\underline{\pi}(H)$ and let $\mathbb{P}^* \in \mathcal{M}$. Then in combination with (3.1) and (i) we have for all $t = 0, \ldots, T$ that

$$\underline{\pi}(H) = E^{\mathbb{P}^*}[H_{\hat{\tau}}] = E^{\mathbb{P}^*}[U^{\downarrow}_{\hat{\tau}}] \geq E^{\mathbb{P}^*}[U^{\downarrow}_{\hat{\tau}\wedge t}] \geq U^{\downarrow}_0 = \underline{\pi}(H),$$

thus $(U^{\downarrow})^{\hat{\tau}}$ is a martingale under $\mathbb{P}^*$. □

We denote by $\mathcal{T}^*$ the set of all optimal stopping times:

$$\mathcal{T}^* := \{\tau \in \mathcal{T} \mid \tau \text{ is an optimal stopping for } H \text{ under some } \mathbb{Q} \in \mathcal{M}\}.$$

**Lemma 3.5.** *Let $\tau \in \mathcal{T}$ be such that $H_\tau$ is replicable, then the unique arbitrage-free price $p$ of $H_\tau$ satisfies $p \leq \underline{\pi}(H)$. Moreover, if $\tau \in \mathcal{T}^*$, then $p = \underline{\pi}(H)$.*

*Proof.* For any $\tau \in \mathcal{T}$ and $\mathbb{Q} \in \mathcal{M}$ we have

$$p = E^{\mathbb{Q}}[H_\tau] \leq \sup_{\sigma \in \mathcal{T}} E^{\mathbb{Q}}[H_\sigma] = U_0^{\mathbb{Q}}, \tag{3.2}$$

and taking the infimum on the right-hand side over all $\mathbb{Q} \in \mathcal{M}$ yields $p \leq \underline{\pi}(H)$. Moreover, if $\tau \in \mathcal{T}^*$, then there exists a $\mathbb{Q} \in \mathcal{M}$ such that equality holds in (3.2). □

**Proposition 3.6.** *Let $H_{\hat{\tau}}$ be replicable at price $\underline{\pi}(H)$. Then*

$$\begin{aligned} \mathcal{Q} &:= \left\{ \mathbb{Q} \in \mathcal{M} \mid U_{\hat{\tau}}^{\mathbb{Q}} = H_{\hat{\tau}} \right\} \\ &= \left\{ \mathbb{Q} \in \mathcal{M} \mid U_0^{\mathbb{Q}} = \underline{\pi}(H) \right\}. \end{aligned} \tag{3.3}$$

*Proof.* Let $\mathbb{Q} \in \mathcal{Q}$. Since $H_{\hat{\tau}}$ is replicable at price $\underline{\pi}(H)$, according to Proposition 3.4 $(U^{\downarrow})^{\hat{\tau}}$ is a $\mathcal{M}$-martingale, so in particular a $\mathbb{Q}$-martingale. We show that the process

$$\widetilde{U}_t := U_t^{\mathbb{Q}} 1_{\{\hat{\tau} < t\}} + U_t^{\downarrow} 1_{\{\hat{\tau} \geq t\}}$$

is a $\mathbb{Q}$-supermartingale dominating $H$. Indeed, for any $t \in \{1, \ldots T\}$ we have that

$$\begin{aligned} E^{\mathbb{Q}}[\widetilde{U}_t \mid \mathcal{F}_{t-1}] &= E^{\mathbb{Q}}[U_t^{\mathbb{Q}} \mid \mathcal{F}_{t-1}] 1_{\{\hat{\tau} < t\}} + E^{\mathbb{Q}}[U^{\downarrow}_{\hat{\tau}\wedge t} \mid \mathcal{F}_{t-1}] 1_{\{\hat{\tau} \geq t\}} \\ &\leq U_{t-1}^{\mathbb{Q}} 1_{\{\hat{\tau} \leq t-1\}} + U^{\downarrow}_{\hat{\tau}\wedge(t-1)} 1_{\{\hat{\tau} > t-1\}} \\ &= U_{t-1}^{\mathbb{Q}} 1_{\{\hat{\tau} < t-1\}} + U^{\downarrow}_{\hat{\tau}\wedge(t-1)} 1_{\{\hat{\tau} \geq t-1\}} \quad = \quad \widetilde{U}_{t-1}, \end{aligned}$$

where we use the supermartingale property of $U^{\mathbb{Q}}$ and $(U^{\downarrow})^{\hat{\tau}}$ and the fact that $U_{\hat{\tau}}^{\mathbb{Q}} = H_{\hat{\tau}} = U_{\hat{\tau}}^{\downarrow}$ by (3.1). Therefore $\widetilde{U}$ is a $\mathbb{Q}$-supermartingale which obviously dominates $H$ since both $U^{\mathbb{Q}}$ and $U^{\downarrow}$ do. By [2, Proposition 6.11], $U^{\mathbb{Q}}$ is the smallest $\mathbb{Q}$-supermartingale dominating $H$, which implies that $U_t^{\mathbb{Q}} \leq \widetilde{U}_t$ for all $t = 0, \ldots, T$, and thus $U_0^{\mathbb{Q}} \leq \widetilde{U}_0 = \underline{\pi}(H)$. Hence $U_0^{\mathbb{Q}} = \underline{\pi}(H)$, and the inclusion '$\subseteq$' in (3.3) is proved.

Now let $\mathbb{Q} \in \mathcal{M}$ be such that $U_0^{\mathbb{Q}} = \underline{\pi}(H)$. Then, as $U^{\mathbb{Q}}$ is a $\mathbb{Q}$-supermartingale dominating $H$ and as $H_{\hat{\tau}}$ is replicable at price $\underline{\pi}(H)$, we have

$$\underline{\pi}(H) = U_0^{\mathbb{Q}} \geq E^{\mathbb{Q}}[U_{\hat{\tau}}^{\mathbb{Q}}] \geq E^{\mathbb{Q}}[H_{\hat{\tau}}] = \underline{\pi}(H).$$

This implies $U_{\hat{\tau}}^{\mathbb{Q}} = H_{\hat{\tau}}$ and concludes the proof of the proposition. □

Our main result Theorem 2.8 follows from the subsequent Theorem 3.7, in which we give equivalent conditions characterizing the case $\underline{\pi}(H) \in \Pi(H)$.

**Theorem 3.7.** *The following conditions are equivalent:*

(i) $\underline{\pi}(H) \in \Pi(H)$.

(ii) $H_{\hat{\tau}}$ *is replicable.*

(iii) $H_{\hat{\tau}}$ *is replicable at price* $\underline{\pi}(H)$.

(iv) *There exists* $\mathbb{Q} \in \mathcal{M}$ *such that* $U_0^{\mathbb{Q}} = \underline{\pi}(H)$.

(v) *There exists* $\tau \in \mathcal{T}^*$ *such that* $H_\tau$ *is replicable.*

*Proof.* (i) ⇒ (iii): Let $\underline{\pi}(H) \in \Pi(H)$. The second property of Definition 2.5 implies the existence of some $\widetilde{\mathbb{P}} \in \mathcal{M}$ such that $\underline{\pi}(H) \geq E^{\widetilde{\mathbb{P}}}[H_{\hat{\tau}}]$. From Proposition 3.4 (i) we know that $(U^{\downarrow})^{\hat{\tau}}$ is a $\mathcal{M}$-submartingale. In conjunction with (3.1) we obtain for all $\mathbb{Q} \in \mathcal{M}$ that

$$E^{\mathbb{Q}}\left[H_{\hat{\tau}}\right] = E^{\mathbb{Q}}\left[U_{\hat{\tau}}^{\downarrow}\right] \geq U_0^{\downarrow} = \underline{\pi}(H).$$

Taking the infimum over all $\mathbb{Q} \in \mathcal{M}$ we arrive at

$$E^{\widetilde{\mathbb{P}}}[H_{\hat{\tau}}] \leq \underline{\pi}(H) \leq \inf_{\mathbb{Q}\in\mathcal{M}} E^{\mathbb{Q}}[H_{\hat{\tau}}] \leq E^{\widetilde{\mathbb{P}}}[H_{\hat{\tau}}],$$

which yields

$$E^{\widetilde{\mathbb{P}}}[H_{\hat{\tau}}] = \underline{\pi}(H) = \inf_{\mathbb{Q}\in\mathcal{M}} E^{\mathbb{Q}}[H_{\hat{\tau}}].$$

Consequently, the set of arbitrage-free prices for the European contingent claim $H_{\hat{\tau}}$ contains its lower bound. Thus $H_{\hat{\tau}}$ is replicable and $\Pi(H_{\hat{\tau}}) = \{\underline{\pi}(H)\}$.

(iii) ⇒ (ii) is obvious.

(ii) ⇒ (iii): Since $H_{\hat{\tau}}$ is replicable, by Lemma 3.5 we have that its price satisfies $p \leq \underline{\pi}(H)$. Now fix $\mathbb{P}^* \in \mathcal{M}$. By the same arguments as in the proof of Proposition 3.4, there is a sequence $(\mathbb{Q}_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ such that $\mathbb{Q}_k|_{\mathcal{F}_{\hat{\tau}}} = \mathbb{P}^*_{\mathcal{F}_{\hat{\tau}}}$ and that

$$U_{\hat{\tau}}^{\mathbb{Q}_k} \searrow U_{\hat{\tau}}^{\downarrow} = H_{\hat{\tau}}.$$

The dominated convergence theorem ensures that

$$\underline{\pi}(H) \leq \lim_{k\to\infty} U_0^{\mathbb{Q}_k} = \lim_{k\to\infty} E^{\mathbb{Q}_k}[U_{\hat{\tau}}^{\mathbb{Q}_k}] = \lim_{k\to\infty} E^{\mathbb{P}^*}[U_{\hat{\tau}}^{\mathbb{Q}_k}] = E^{\mathbb{P}^*}[H_{\hat{\tau}}] = p,$$

where we use that $(U^{\mathbb{Q}_k})^{\tau^{\mathbb{Q}_k}}$ is a $\mathbb{Q}_k$-martingale, $\hat{\tau} \le \tau^{\mathbb{Q}_k}$, and $\mathbb{Q}_k|_{\mathcal{F}_{\hat{\tau}}} = \mathbb{P}^*|_{\mathcal{F}_{\hat{\tau}}}$.
(iii) $\Rightarrow$ (iv): Fix $\mathbb{P}^* \in \mathcal{M}$. According to Lemma 3.2, there is a sequence $(\mathbb{Q}_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ such that $A_k := \{\tau^{\mathbb{Q}_k} = \hat{\tau}\} \nearrow \Omega$. Again we may assume that $\mathbb{Q}_k|_{\mathcal{F}_{\hat{\tau}}} = \mathbb{P}^*_{\mathcal{F}_{\hat{\tau}}}$. Define

$$B_k := A_k \setminus \bigcup_{m=1}^{k-1} A_m \in \mathcal{F}_{\hat{\tau}}.$$

Then

$$\hat{\tau} = \sum_{k=1}^{\infty} \tau^{\mathbb{Q}_k} 1_{B_k}.$$

Now consider the probability measure $\widetilde{\mathbb{P}}$ obtained by pasting the measure $\mathbb{P}^*$ with the measures $\mathbb{Q}_k$ on $B_k$ in $\hat{\tau}$, i.e., $\widetilde{\mathbb{P}}$ defined via

$$\widetilde{\mathbb{P}}(A) = E^{\mathbb{P}^*}\left[\sum_{k=1}^{\infty} E^{\mathbb{Q}_k}[1_{A\cap B_k} \mid \mathcal{F}_{\hat{\tau}}]\right], \quad A \in \mathcal{F}.$$

Clearly $\widetilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$. Moreover, $\widetilde{\mathbb{P}} \in \mathcal{M}$ since for $i = 1, \dots, d$ and $t = 0, \dots, T-1$ we have

$$\begin{aligned} E^{\widetilde{\mathbb{P}}}[S^i_{t+1} \mid \mathcal{F}_t] &= E^{\mathbb{P}^*}[S^i_{t+1} \mid \mathcal{F}_t]1_{\{\hat{\tau}\ge t+1\}} + \sum_{k=1}^{\infty} E^{\mathbb{Q}_k}[S^i_{t+1}1_{B_k} \mid \mathcal{F}_t]1_{\{\hat{\tau}\le t\}} \\ &= E^{\mathbb{P}^*}[S^i_{t+1} \mid \mathcal{F}_t]1_{\{\hat{\tau}\ge t+1\}} + \sum_{k=1}^{\infty} E^{\mathbb{Q}_k}[S^i_{t+1} \mid \mathcal{F}_t]1_{B_k\cap\{\hat{\tau}\le t\}} \quad = \quad S^i_t \end{aligned}$$

as $B_k \cap \{\hat{\tau} \le t\} \in \mathcal{F}_t$. Since on $B_k$ we have $U^{\mathbb{Q}_k}_{\hat{\tau}} = H_{\hat{\tau}}$, by monotone convergence

$$\begin{aligned} H_{\hat{\tau}} &= \sum_{k=1}^{\infty} U^{\mathbb{Q}_k}_{\hat{\tau}} 1_{B_k} \quad = \quad \sum_{k=1}^{\infty} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T},\sigma\ge\hat{\tau}} E^{\mathbb{Q}_k}[H_\sigma 1_{B_k} \mid \mathcal{F}_{\hat{\tau}}] \\ &= \sum_{k=1}^{\infty} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T},\sigma\ge\hat{\tau}} E^{\widetilde{\mathbb{P}}}[H_\sigma 1_{B_k} \mid \mathcal{F}_{\hat{\tau}}] \quad \ge \quad \operatorname*{ess\,sup}_{\sigma\in\mathcal{T},\sigma\ge\hat{\tau}} \sum_{k=1}^{\infty} E^{\widetilde{\mathbb{P}}}[H_\sigma 1_{B_k} \mid \mathcal{F}_{\hat{\tau}}] \\ &= \operatorname*{ess\,sup}_{\sigma\in\mathcal{T},\sigma\ge\hat{\tau}} E^{\widetilde{\mathbb{P}}}[H_\sigma \mid \mathcal{F}_{\hat{\tau}}] \quad = \quad U^{\widetilde{\mathbb{P}}}_{\hat{\tau}} \\ &\ge H_{\hat{\tau}}. \end{aligned}$$

This means that $\widetilde{\mathbb{P}} \in \mathcal{M}$ verifies $U^{\widetilde{\mathbb{P}}}_{\hat{\tau}} = H_{\hat{\tau}}$. Since $H_{\hat{\tau}}$ is replicable at price $\underline{\pi}(H)$, by Proposition 3.6 we obtain that $U^{\widetilde{\mathbb{P}}}_0 = \underline{\pi}(H)$.
(iv) $\Rightarrow$ (i): As already mentioned, $U^{\mathbb{Q}}_0 = E^{\mathbb{Q}}[H_{\tau^{\mathbb{Q}}}]$ clearly satisfies both conditions in Definition 2.5.
(iv) $\Rightarrow$ (v): If there is $\mathbb{Q} \in \mathcal{M}$ such that $U^{\mathbb{Q}}_0 = \underline{\pi}(H)$, then, according to the equivalences already proved, $H_{\hat{\tau}}$ is replicable at price $\underline{\pi}(H)$. This yields

$$\underline{\pi}(H) = U^{\mathbb{Q}}_0 = E^{\mathbb{Q}}\left[H_{\tau^{\mathbb{Q}}}\right] \ge E^{\mathbb{Q}}\left[H_{\hat{\tau}}\right] = \underline{\pi}(H).$$

Consequently $E^{\mathbb{Q}}\left[H_{\tau^{\mathbb{Q}}}\right] = E^{\mathbb{Q}}\left[H_{\hat{\tau}}\right]$, which means that $\hat{\tau}$ is optimal under $\mathbb{Q}$ and consequently $\tau^{\mathbb{Q}} = \hat{\tau}$, since $\hat{\tau} \le \tau^{\mathbb{Q}}$ and by minimality of $\tau^{\mathbb{Q}}$. This implies that $H_{\tau^{\mathbb{Q}}} = H_{\hat{\tau}}$ is replicable.

(v) $\Rightarrow$ (iv): Let $\mathbb{Q} \in \mathcal{M}$ and $\tau$ be an optimal stopping time under $\mathbb{Q}$, with $H_\tau$ being replicable. Then, according to Lemma 3.5, the unique arbitrage-free price of $H_\tau$ is $\underline{\pi}(H)$, hence

$$U_0^{\mathbb{Q}} = E^{\mathbb{Q}}[H_\tau] = \underline{\pi}(H).$$

□

Notice that Theorem 3.7 extends the case of European contingent claims. Indeed, let $Y$ be a discounted European contingent claim and $H$ the corresponding American contingent claim defined in (2.2). Then clearly $H_{\hat{\tau}} = Y$, thus $\underline{\pi}(H) = \inf \Pi(Y)$ is arbitrage-free if and only if $Y$ is replicable.

**Remark 3.8.** A major source of ideas how to attack the problem which we consider is [9], where a problem analogous to (2.5) is studied in a continuous-time setting. More precisely, [9] investigates the existence of a "worst-case probability measure" $\mathbb{Q}$ for the lower Snell envelope of an American option $H$ with respect to a convex family $\mathcal{N}$ of equivalent probability measures, in the sense that $\mathbb{Q} \in \mathcal{N}$ shall satisfy

$$\sup_{\tau \in \mathcal{T}} E^{\mathbb{Q}}[H_\tau] = \inf_{P \in \mathcal{N}} \sup_{\tau \in \mathcal{T}} E^P[H_\tau].$$

It is shown that such a measure $\mathbb{Q}$ exists under a compactness assumption on the set of densities $\left\{\frac{dP}{d\mathbb{P}} \mid P \in \mathcal{N}\right\}$; see also [7], where the multiple prior Snell envelope is studied under the same assumption. However, when pricing an American contingent claim in a financial market, the set of test measures $\mathcal{N}$ equals the set of equivalent martingale measures $\mathcal{M}$, for which this compactness assumption is satisfied if and only if the market is complete ($\mathcal{M} = \{\mathbb{Q}\}$). Hence the approach of [9] does not suite our purposes. Note that Theorem 3.7 does not require any further condition on the set of equivalent martingale measures $\mathcal{M}$. Moreover, notice that our results are easily extended to continuous-time financial markets in case the American contingent claim has a discrete tenor structure. ◊

Our main results are expressed in terms of the stopping time $\hat{\tau}$, for which we know that $U^{\downarrow}_{\hat{\tau}} = H_{\hat{\tau}}$; see (3.1). Let us consider the first time when the lower Snell envelope $U^{\downarrow}$ of $H$ equals $H$, that is,

$$\tau^{\downarrow} := \inf\{t \geq 0 \mid U^{\downarrow}_t = H_t\}.$$

Clearly we have $\tau^{\downarrow} \leq \hat{\tau}$. It might be expected that $\tau^{\downarrow}$ plays a similarly important role in the analysis of $U^{\downarrow}$ as the stopping times $\tau^{\mathbb{Q}}$ do for $U^{\mathbb{Q}}$. Concerning this matter, see for instance the discussion of the lower Snell envelope as outlined in [2]. It follows directly from Proposition 3.4 and Doob's stopping theorem that $U^{\downarrow}$ stopped at $\tau^{\downarrow}$ is a $\mathcal{M}$-submartingale, and a $\mathcal{M}$-martingale in case $\underline{\pi}(H)$ is arbitrage-free. Hence, a natural question is whether $\tau^{\downarrow}$ and $\hat{\tau}$ always coincide, or in case they do not, whether at least the analysis carried out in this section could also be done replacing $\hat{\tau}$ by the earlier stopping time $\tau^{\downarrow}$. However, the answer to both questions is no. In Example 4.1 we show that $\tau^{\downarrow}$ and $\hat{\tau}$ need not coincide, and that $H_{\tau^{\downarrow}}$ can be replicable without $\underline{\pi}(H)$ being an arbitrage-free price for $H$. Consequently, $\tau^{\downarrow}$ is not suited for a characterization of the situation $\underline{\pi}(H) \in \Pi(H)$. Nevertheless, if $\underline{\pi}(H) \in \Pi(H)$, then in particular $\hat{\tau} = \tau^{\downarrow}$ as it is shown in the following Proposition 3.9. To this end, note that

$$\tau^{\downarrow} \in \mathcal{T}^* \quad \Longleftrightarrow \quad \tau^{\downarrow} = \hat{\tau} = \tau^{\mathbb{Q}} \quad \text{for some} \quad \mathbb{Q} \in \mathcal{M}.$$

**Proposition 3.9.** *The following conditions are equivalent:*

(i) $\underline{\pi}(H) \in \Pi(H)$,

(ii) $\tau^{\downarrow} \in \mathcal{T}^*$ *and* $H_{\tau^{\downarrow}}$ *is replicable.*

*Proof.* Suppose that $\underline{\pi}(H) \in \Pi(H)$. Then, according to Theorem 3.7, $H_{\hat{\tau}}$ is replicable at price $\underline{\pi}(H)$. Hence Proposition 3.4 implies that $(U^{\downarrow})^{\hat{\tau}}$ is a $\mathcal{M}$-martingale. Now Doob's stopping theorem yields

$$E^{\mathbb{Q}}[H_{\tau^{\downarrow}}] = E^{\mathbb{Q}}[U^{\downarrow}_{\tau^{\downarrow}}] = U^{\downarrow}_0 = \underline{\pi}(H) \quad \text{for all } \mathbb{Q} \in \mathcal{M}.$$

In other words, $H_{\tau^{\downarrow}}$ has a unique arbitrage-free price given by $\underline{\pi}(H)$ and is thus replicable. Moreover, in Theorem 3.7 it is shown that there exists a measure $\mathbb{Q} \in \mathcal{M}$ such that $H_{\tau^{\mathbb{Q}}}$ is replicable at price $\underline{\pi}(H)$. This means that $E^{\mathbb{Q}}[H_{\tau^{\mathbb{Q}}}] = \underline{\pi}(H) = E^{\mathbb{Q}}[H_{\tau^{\downarrow}}]$, so $\tau^{\downarrow}$ is optimal for $\mathbb{Q}$. By minimality of $\tau^{\mathbb{Q}}$, and since $\tau^{\downarrow} \leq \hat{\tau} \leq \tau^{\mathbb{Q}}$, it follows that $\tau^{\downarrow} = \hat{\tau} = \tau^{\mathbb{Q}}$.

The reverse implication follows directly from Theorem 3.7. □

# 4 An Illustrating Example

In Example 4.1 we show how in a given incomplete market, in the case of non-replicable American contingent claims $H$, we may encounter both $\underline{\pi}(H) \in \Pi(H)$ and $\underline{\pi}(H) \notin \Pi(H)$; see also [2, Example 6.34].

**Example 4.1.** Let $X_1, X_2$ be standard normal distributed random variables on the probability spaces $(\Omega_i, \mathcal{A}_i, \mathbb{P}_i)$ respectively, and consider the product space $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{A}_1 \otimes \mathcal{A}_2$, and $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$. We define the random variables $\widetilde{X}_i$ on $(\Omega, \mathcal{F}, \mathbb{P})$ by $\widetilde{X}_i(\omega_1, \omega_2) = -1 + \sqrt{2} X_i(\omega_i), i = 1, 2$. Let the discounted stock price of the risky asset on $(\Omega, \mathcal{F}, \mathbb{P})$ be given by

$$S_0 = 1, \quad S_1 = e^{\widetilde{X}_1}, \quad S_2 = e^{\widetilde{X}_1 + \widetilde{X}_2}.$$

The filtration is

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_1 = \sigma(\widetilde{X}_1), \quad \mathcal{F}_2 = \sigma(\widetilde{X}_1, \widetilde{X}_2).$$

Consider the following discounted American contingent claim:

$$H_0 = 0, \quad H_1 = e^{\widetilde{X}_1}, \quad H_2 = e^{\widetilde{X}_1 + \frac{1}{2}\widetilde{X}_2}.$$

Clearly $\tau^{\mathbb{Q}} \geq 1$ for any equivalent martingale measure $\mathbb{Q} \in \mathcal{M}$. Moreover, note that $\mathbb{P} \in \mathcal{M}$ and that, for any $\tau \in \mathcal{T}$ such that $\tau \geq 1$,

$$\begin{aligned} E^{\mathbb{P}}[H_\tau] &= E^{\mathbb{P}}[e^{\widetilde{X}_1} 1_{\{\tau=1\}} + e^{\widetilde{X}_1 + \frac{1}{2}\widetilde{X}_2} 1_{\{\tau=2\}}] \\ &= E^{\mathbb{P}}[e^{\widetilde{X}_1} 1_{\{\tau=1\}}] + E^{\mathbb{P}}[e^{\widetilde{X}_1} 1_{\{\tau=2\}}] \cdot E^{\mathbb{P}}[e^{\frac{1}{2}\widetilde{X}_2}] \\ &\leq E^{\mathbb{P}}[e^{\widetilde{X}_1} 1_{\{\tau=1\}}] + E^{\mathbb{P}}[e^{\widetilde{X}_1} 1_{\{\tau=2\}}] = 1, \end{aligned}$$

where the last inequality is strict if $\mathbb{P}(\tau = 2) > 0$ since $E^{\mathbb{P}}[e^{\frac{1}{2}\widetilde{X}_2}] < 1$. In particular this gives $\tau^{\mathbb{P}} = 1$, which in turn implies $\hat{\tau} = 1$. Therefore, $H_{\hat{\tau}} = S_1$ is replicable and Theorem 3.7 ensures that $\underline{\pi}(H)$ is an arbitrage-free price for $H$.

Now consider another discounted American contingent claim, given by

$$H_0 = 0, \quad H_1 = e^{\widetilde{X}_1}, \quad H_2 = e^{\widetilde{X}_1} Z \quad \text{where } Z = e^{\widetilde{X}_2} 1_{\{\widetilde{X}_2 > 1\}} + 1_{\{\widetilde{X}_2 \leq 1\}}.$$

Since $Z \geq 1$ and $\mathbb{P}(Z > 1) > 0$, for each stopping time $\tau \in \mathcal{T}$ we have

$$H_\tau = e^{\widetilde{X}_1} 1_{\{\tau=1\}} + e^{\widetilde{X}_1} Z 1_{\{\tau=2\}} \leq H_2,$$

which implies that $\tau^{\mathbb{Q}} = 2$ for all $\mathbb{Q} \in \mathcal{M}$. Indeed, if for some stopping time $\sigma \in \mathcal{T}$ we have $\mathbb{P}(\sigma = 1) > 0$, then $\{\sigma = 1\} = A \times \Omega_2$ for some $A \in \mathcal{A}_1$ with $\mathbb{P}_1(A) > 0$, because $\mathcal{F}_1 = \sigma\{X_1^{-1}(B) \times \Omega_2 \mid B \in \mathcal{B}(\mathbb{R})\}$. Consequently, $\mathbb{P}(\{\sigma = 1\} \cap \{\widetilde{X}_2 > 1\}) = \mathbb{P}_1(A)\mathbb{P}_2(X_2 > \sqrt{2}) > 0$, and on the set $\{\sigma = 1\} \cap \{\widetilde{X}_2 > 1\}$ the stopped claim $H_\sigma$ is strictly smaller than $H_2$ whereas always $H_\sigma \leq H_2$. Therefore we obtain that $\hat{\tau} = 2$. However, one can find a sequence of equivalent martingale measures $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ such that $E^{\mathbb{Q}_n}[Z \mid \mathcal{F}_1] \to 1$ as $n \to \infty$. Consequently, this yields

$$U_1^{\downarrow} = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}} \operatorname*{ess\,sup}_{\tau\in\mathcal{T},\tau\geq 1} E^{\mathbb{Q}}[H_\tau \mid \mathcal{F}_1] = H_1.$$

Therefore $\tau^{\downarrow} = 1 < 2 = \hat{\tau}$. In addition we have that $H_{\tau^{\downarrow}} = S_1$ is replicable, whereas $H_{\hat{\tau}}$ is not, so $\underline{\pi}(H)$ is not an arbitrage-free price by Theorem 3.7. ♢

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